Tracking Control in The Wasserstein Space

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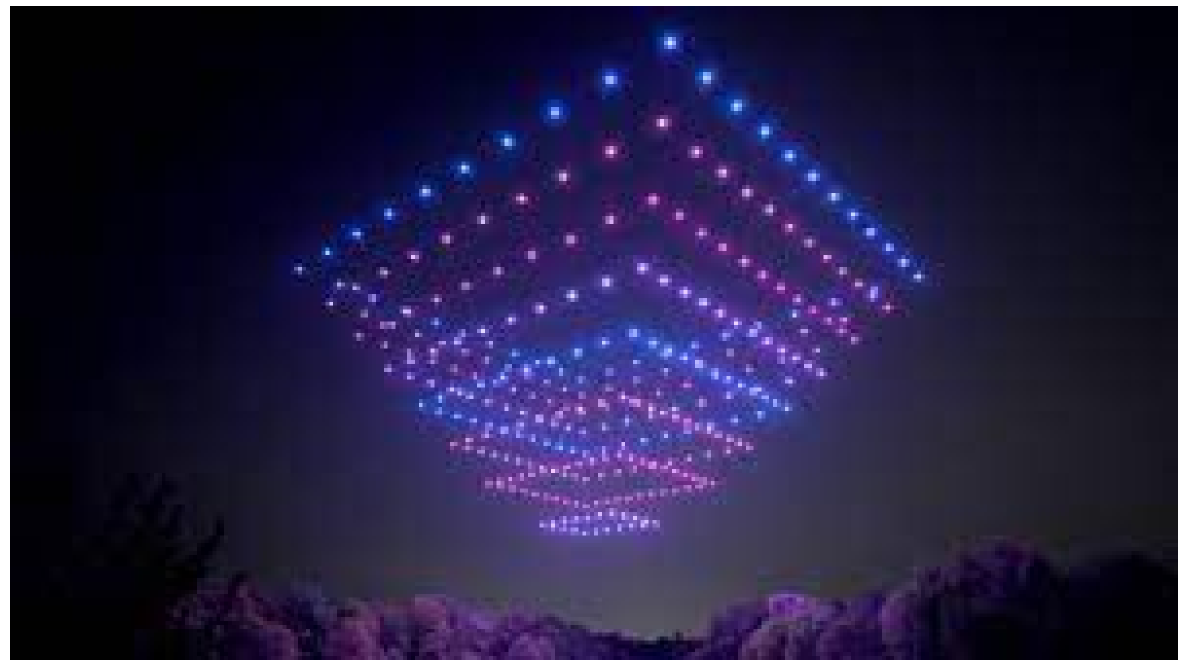
Many Applications for Autonomous Swarms

Emergency Response

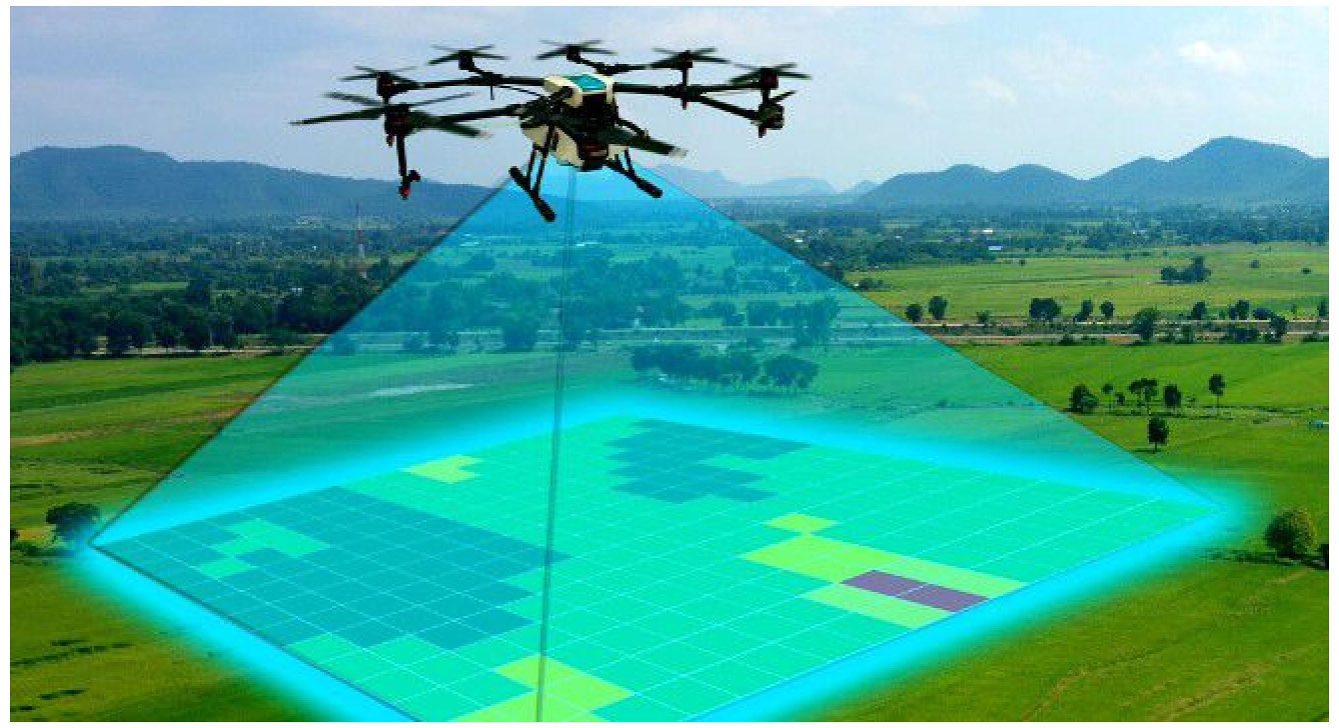
Logistics



Entertainment

Transportation

Defense



Data Collection

Motivation

The Problem in Focus:

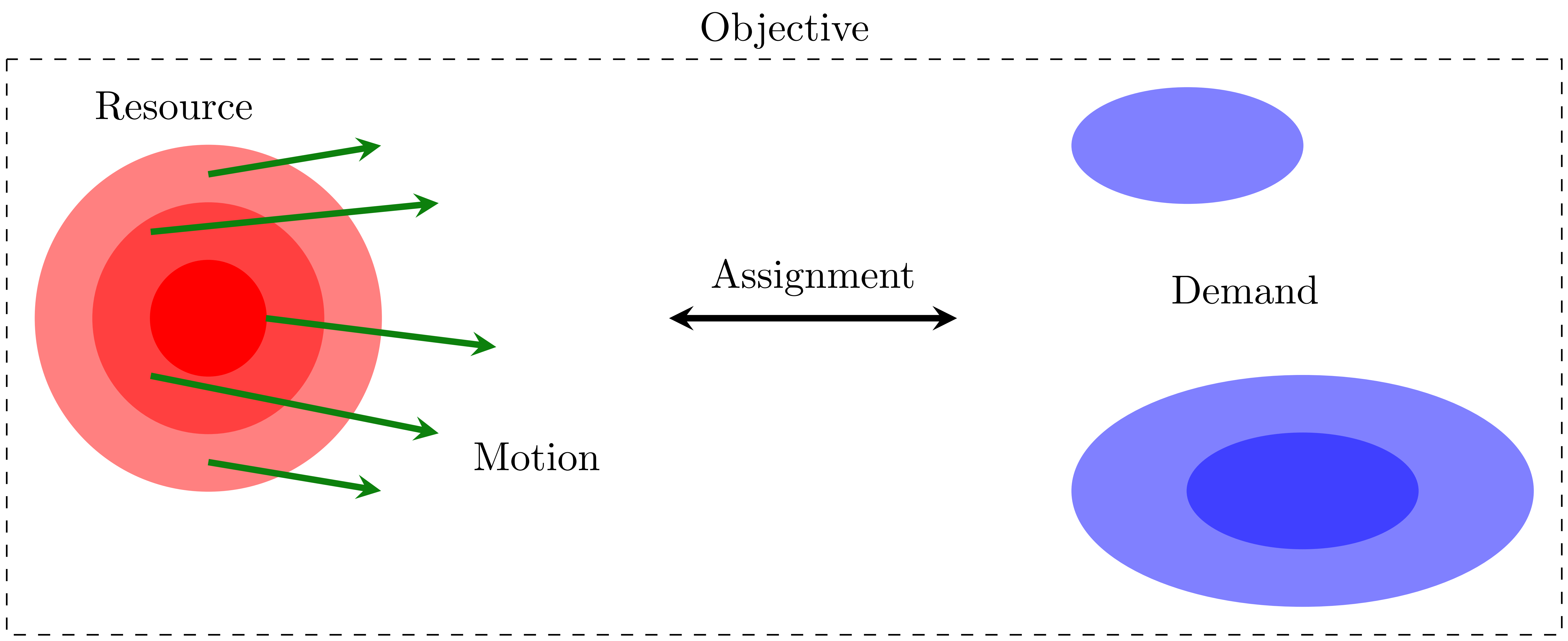
- Large swarms robust/efficient, but hard to model/control
- Want to develop theoretical foundations for design heuristics

Aim to Answer Questions:

- How should large swarms move and communicate?
- Which control architectures can achieve which behaviors?
- What are the attainable performance limits of these architectures?

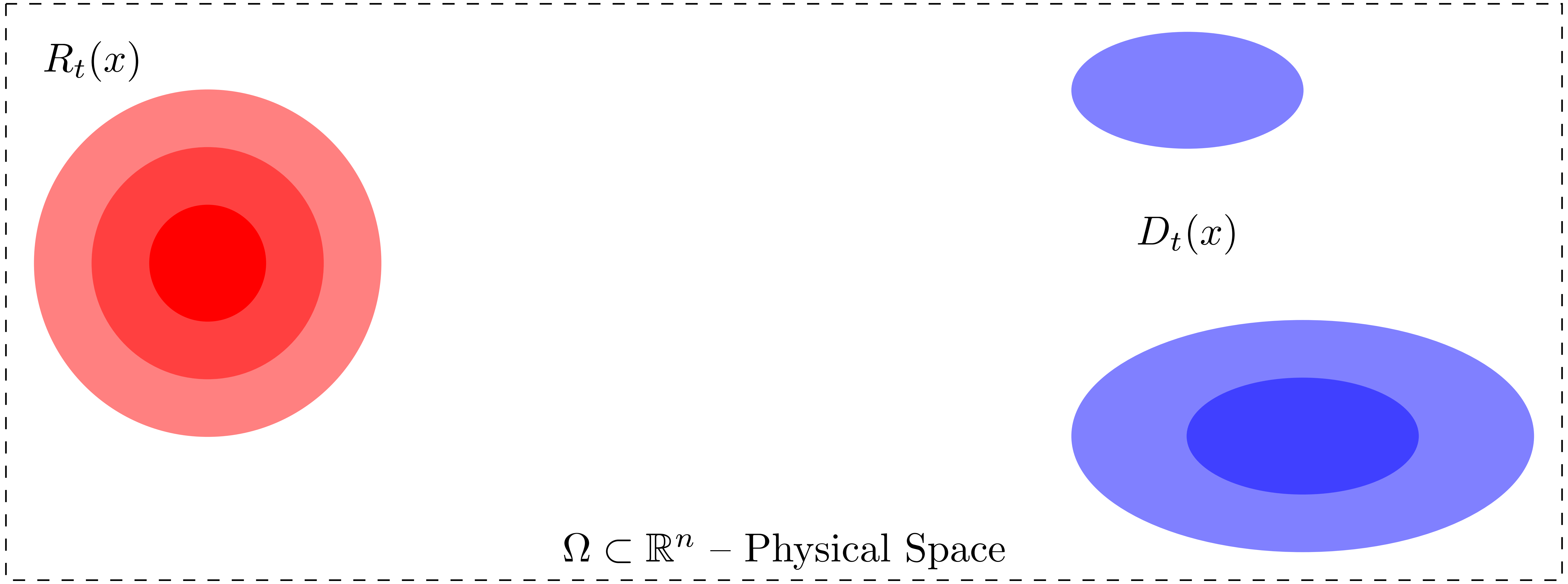
Approach

- This work: what sorts of motion patterns are optimal?
- Looking at motion planning and control for **tracking**
- Using **continuum models, optimal transport, optimal control**



Problem Formulation: Resource/Demand Densities

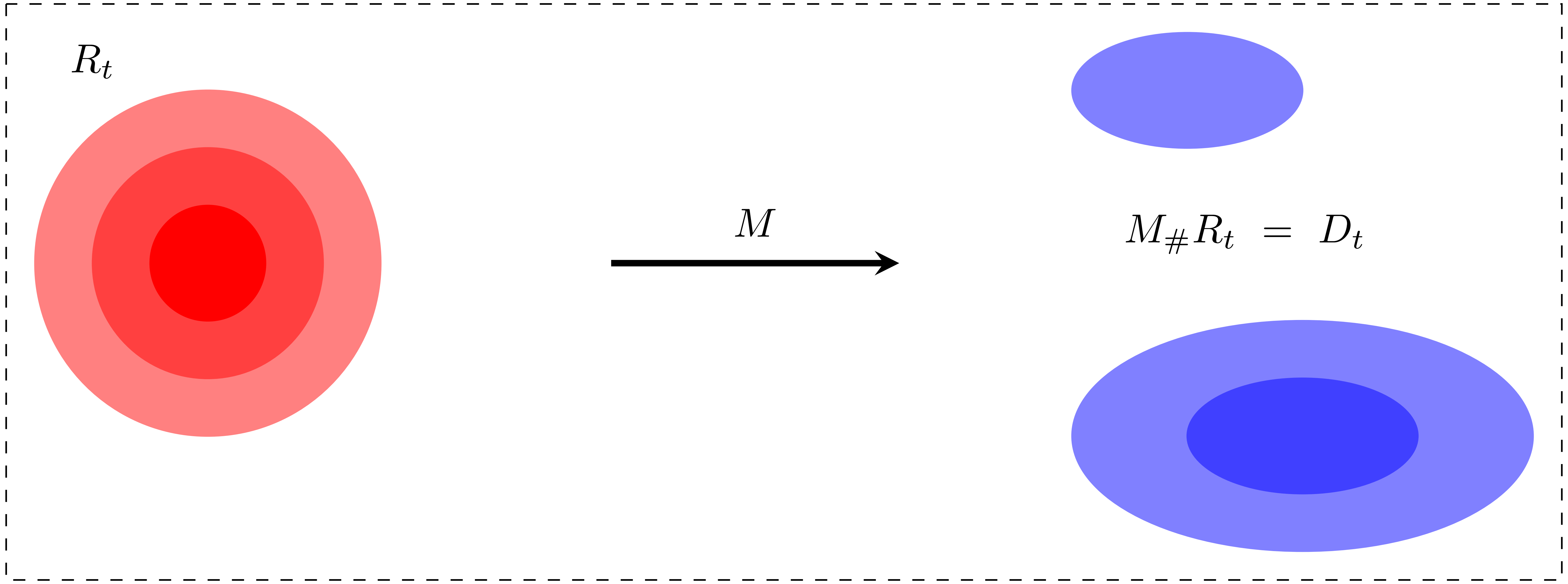
- Resource = controlled mobile agents (provides services)
- Demand = known entity (requires services)



Problem Formulation: Assignment

Monge Problem (Optimal Transport): $\inf \int_\Omega \|M(x) - x\|_2^2 \, R_t(x)\, dx$ s.t. $M_\# R_t = D_t$

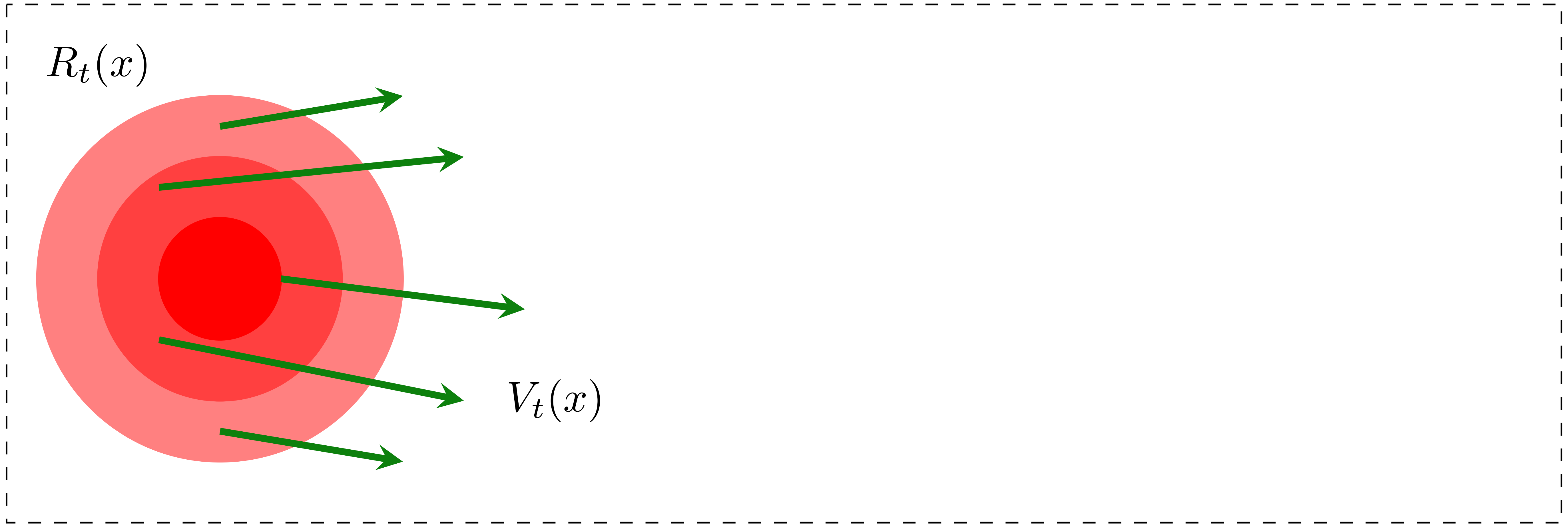
- # denotes **measure pushforward**
- Minimizer $\bar{M}_{R_t \to D_t}$ is **optimal assignment map**
- Minimum $W_2^2(R_t, D_t)$ is **Wasserstein distance**



Problem Formulation: Dynamic Model

- Tracking → want resources close to demand
- Control resource through **velocity field** V

Dynamics (Continuity Equation): $$\partial_t R_t(x) \;=\; -\nabla \cdot \big(R_t(x)\, V_t(x)\big)$$



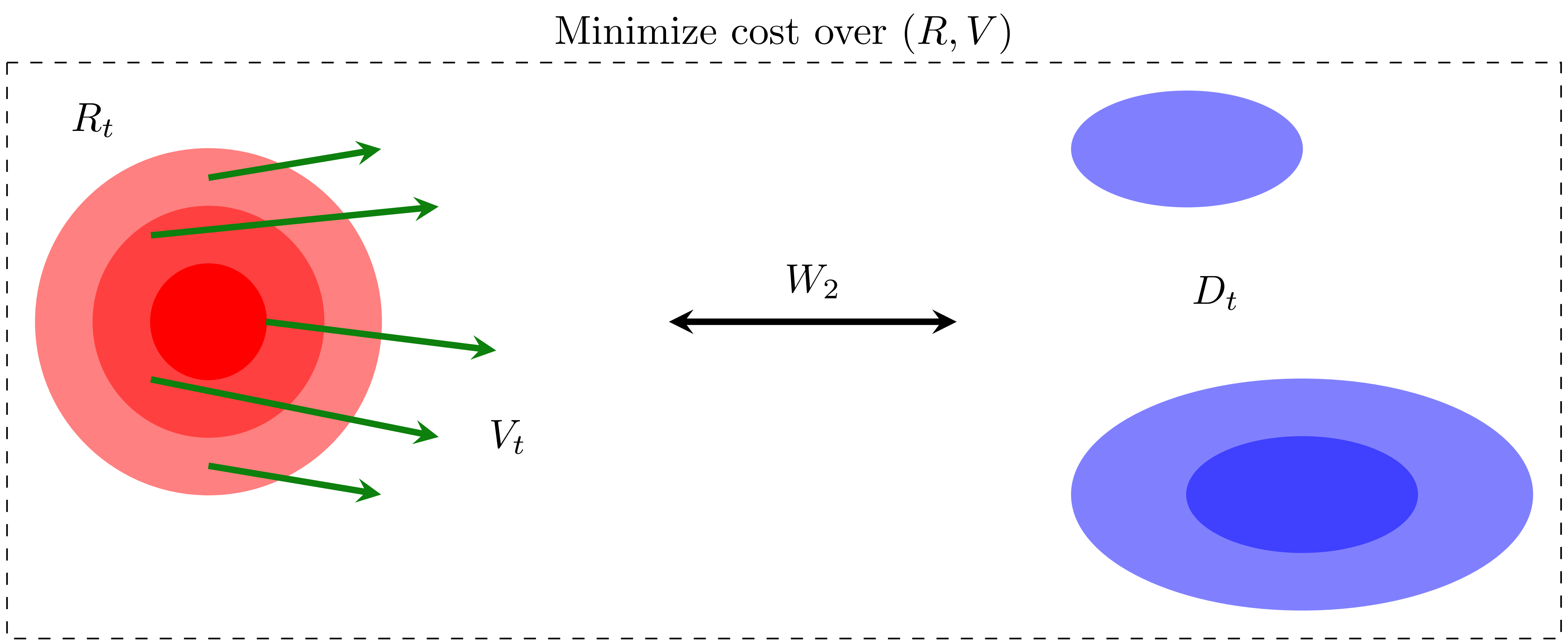
Motion Cost: $$\|V_t\|^2_{L^2(R_t)} \;:=\; \int_\Omega \|V_t(x)\|_2^2\, R_t(x)\, dx$$

Formal Problem Statement

Given an initial resource distribution R_0 and demand trajectory D over $[0,T]$, solve

$$\inf_{R,V} \int_0^T \underbrace{W_2^2(R_t, D_t)}_{\text{Assignment Cost}} + \alpha \underbrace{\|V_t\|^2_{L^2(R_t)}}_{\text{Motion Cost}} dt \qquad \text{s.t.} \qquad \underbrace{\partial_t R_t = -\nabla \cdot (R_t V_t)}_{\text{Dynamic Constraint}}$$

- Intuitively, "R should track D efficiently"
- Trade-off parameter α controls relative importance of costs
- D constant in time → regulation problem



Structural Features of Solution

Necessary Conditions for Optimality:

Optimal velocity field is irrotational!

$$
\begin{aligned}
\partial_t R_t &= -\nabla \cdot (R_t \boxed{\nabla \Lambda_t}) \\
\partial_t \Lambda_t &= -\frac{1}{2}\|\nabla \Lambda_t\|_2^2 + \boxed{\frac{1}{2\alpha}}\boxed{\phi(R_t, D_t)}
\end{aligned}
\qquad
\boxed{\begin{aligned} R_0 &= R_0 \\ \Lambda_T &= 0 \end{aligned}}
$$

Nonlinear two-point boundary value PDE

Demand-tracking forcing term

Requires solving an optimization problem

- Optimal solutions are **noncausal:** need to know D ahead of time
- Computationally expensive
- How to approach this?

Approach and Main Results

- Main tools: Otto calculus, calculus of variations, optimal control

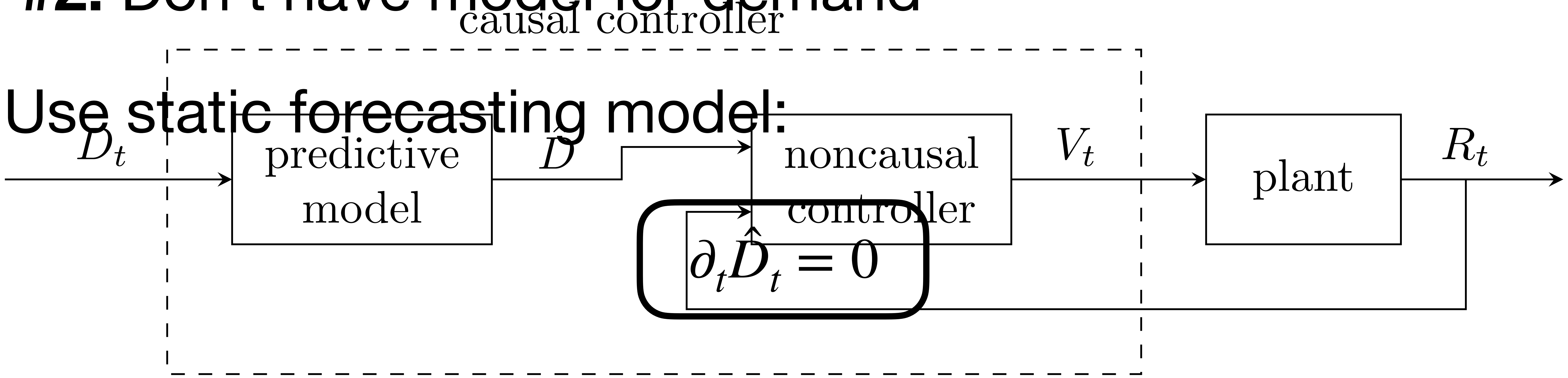
	Regulation	Tracking
1-D	Solutions fully characterized (NecSys ’22)	Solutions fully characterized (TCNS ’25, in review)
n-D	Solutions fully characterized (CDC ’23)	Ongoing work (CDC ’24, …)

Solutions decouple by monotone assignment

Solutions decouple by linearized OT, causal

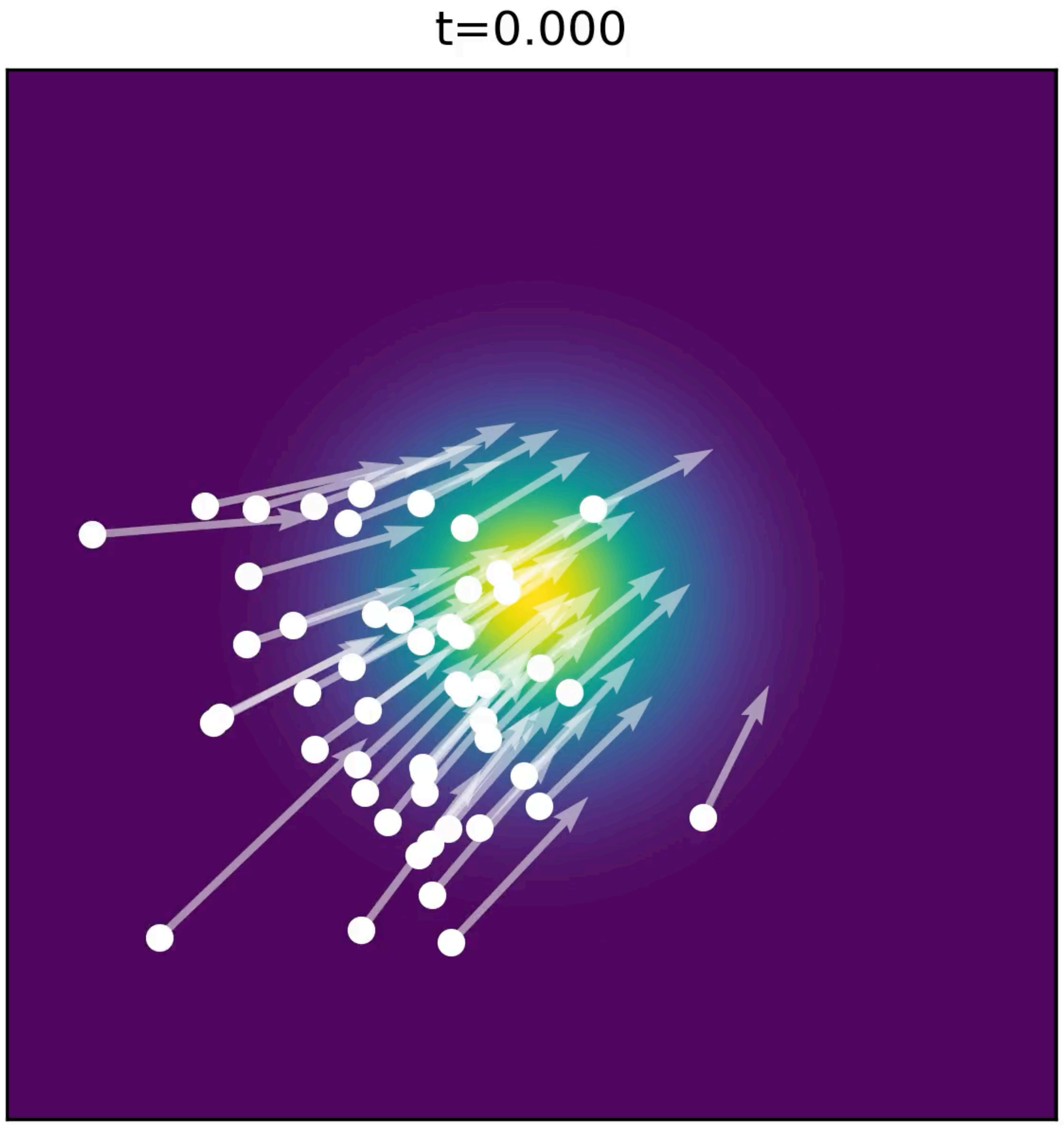
Suboptimal Tracking With Model-Predictive Control

- **Problem #1:** Need to address noncausality
- **Idea #1:** use model to forecast demand trajectory, use forecasted trajectory in necessary conditions in receding horizon scheme
- **Problem #2:** Don't have model for demand
- **Idea #2:** Use static forecasting model:



- (Also solves **Problem #3:** computational cost)

Model-Predictive Control Simulations



Conclusion

Takeaways:

- Simplified models can provide insight and design heuristics
- Leveraging geometric structure can be powerful

Future Work:

- Solving necessary conditions
- More sophisticated demand models
- Investigating resulting controllers

Thanks to My Collaborators

Bassam Bamieh

Stacy Patterson

Jared Jonas

Thanks for Watching! Questions?

Personal Website

Google Scholar