Tracking Control in the Wasserstein Space

44th SoCal Control Workshop
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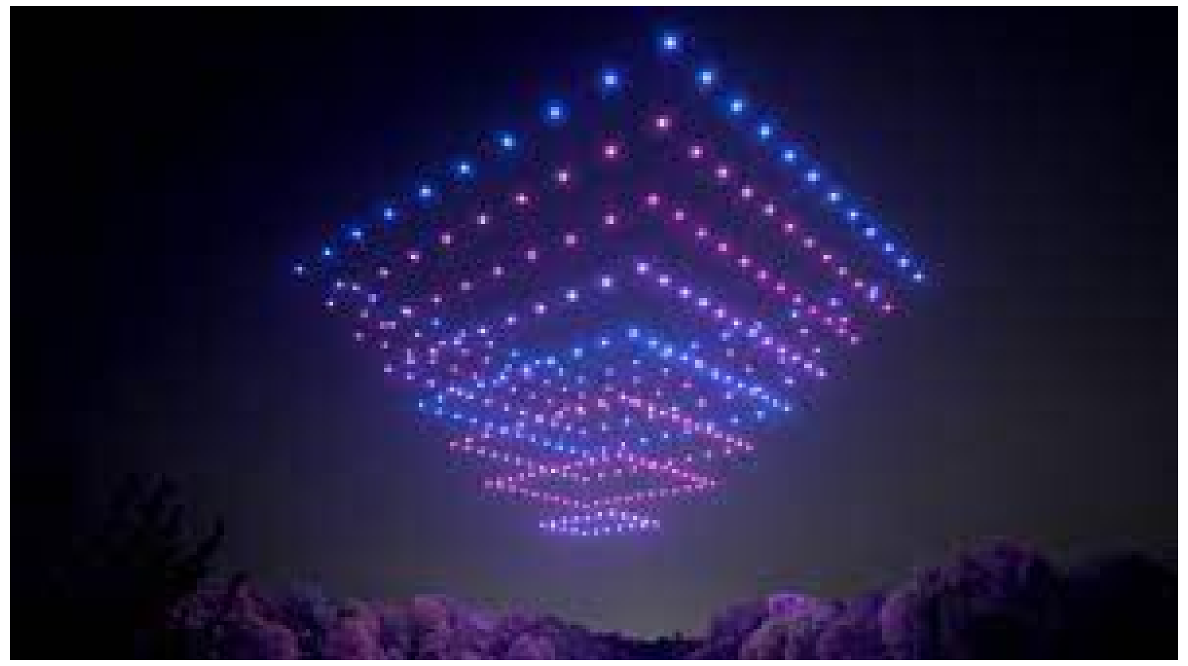
Many Applications for Autonomous Swarms

Emergency Response

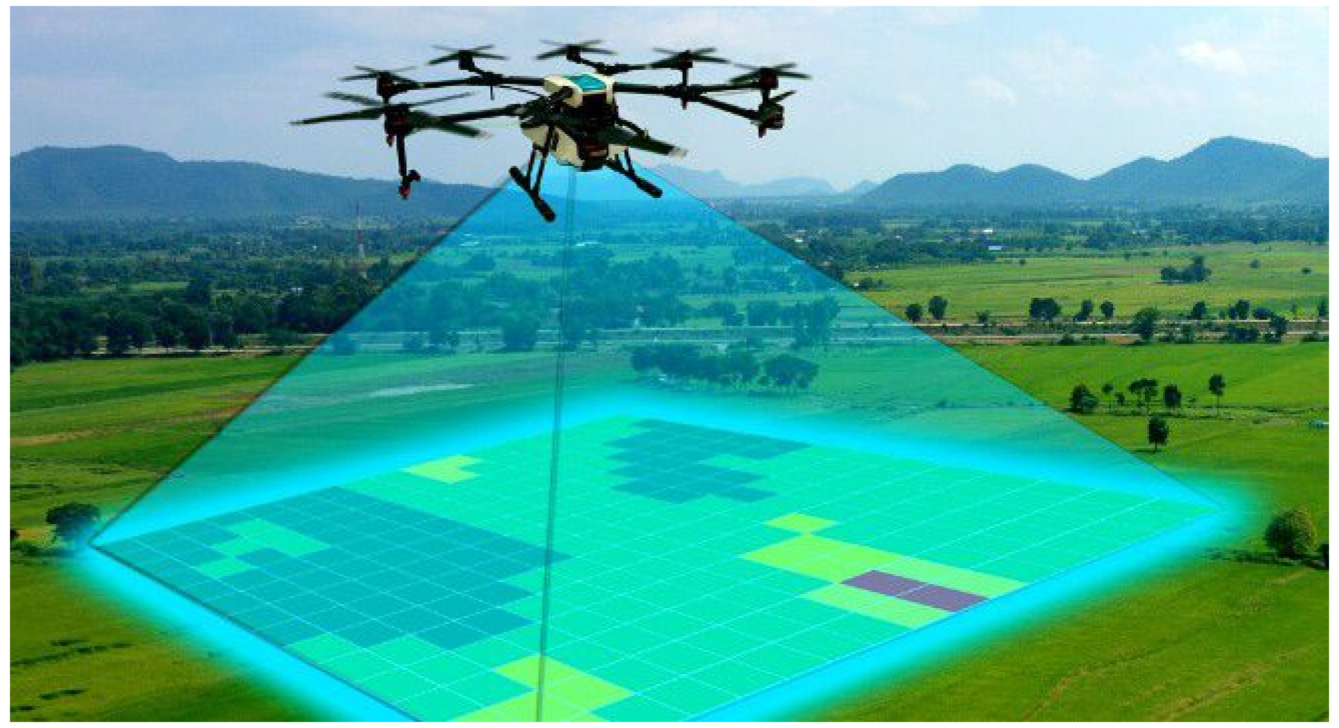
Logistics



Entertainment

Transportation

Defense



Data Collection

Motivation

The Problem in Focus:

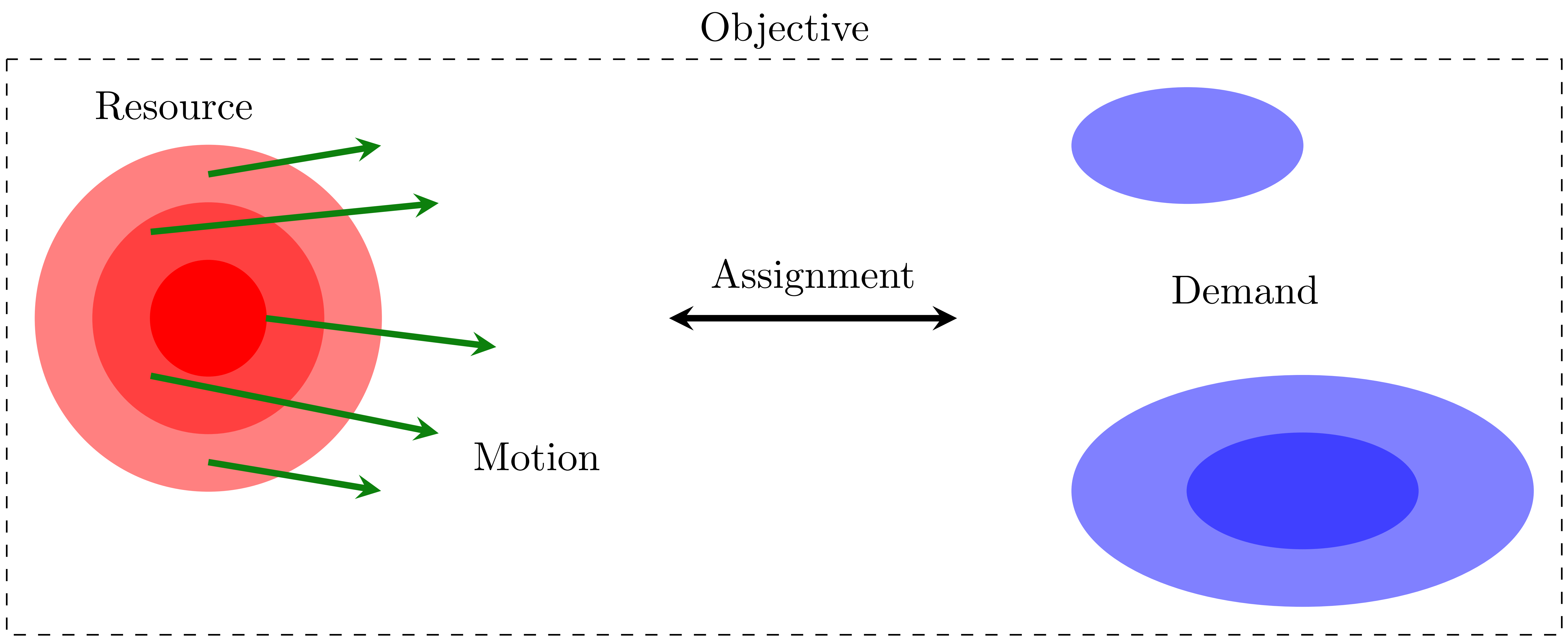
- Large swarms robust/efficient, but hard to model/control
- Want to develop theoretical foundations for design heuristics

Aim to Answer Questions:

- How should large swarms move and communicate?
- Which control architectures can achieve which behaviors?
- What are the attainable performance limits of these architectures?

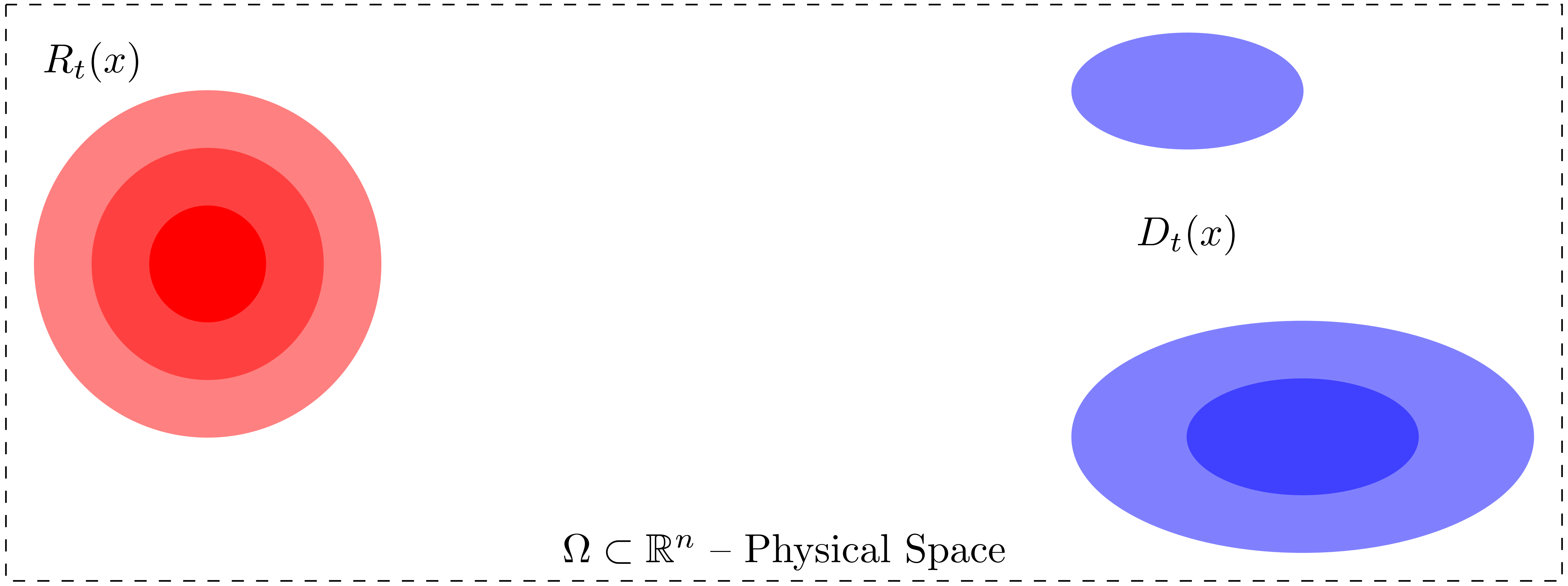
Approach

- This work: what sorts of motion patterns are optimal?
- Looking at motion planning and control for **tracking**
- Using **continuum models, optimal transport, optimal control**



Problem Formulation: Demand/Resource Distributions

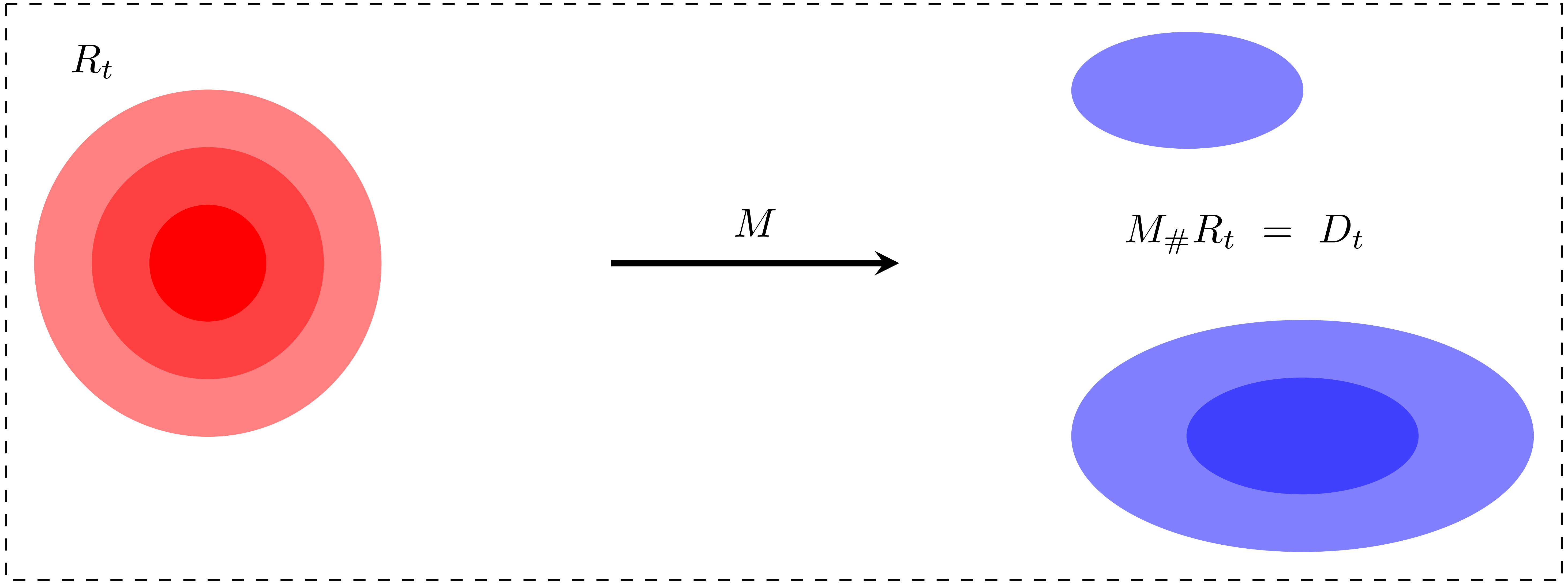
- Demand = known entity (requires services)
- Resource = controlled mobile agents (provides services)



Problem Formulation: Assignment

Monge Problem (Optimal Transport): $\inf \int_\Omega \|M(x) - x\|_2^2 \, R_t(x) \, dx$ s.t. $M_\# R_t = D_t$

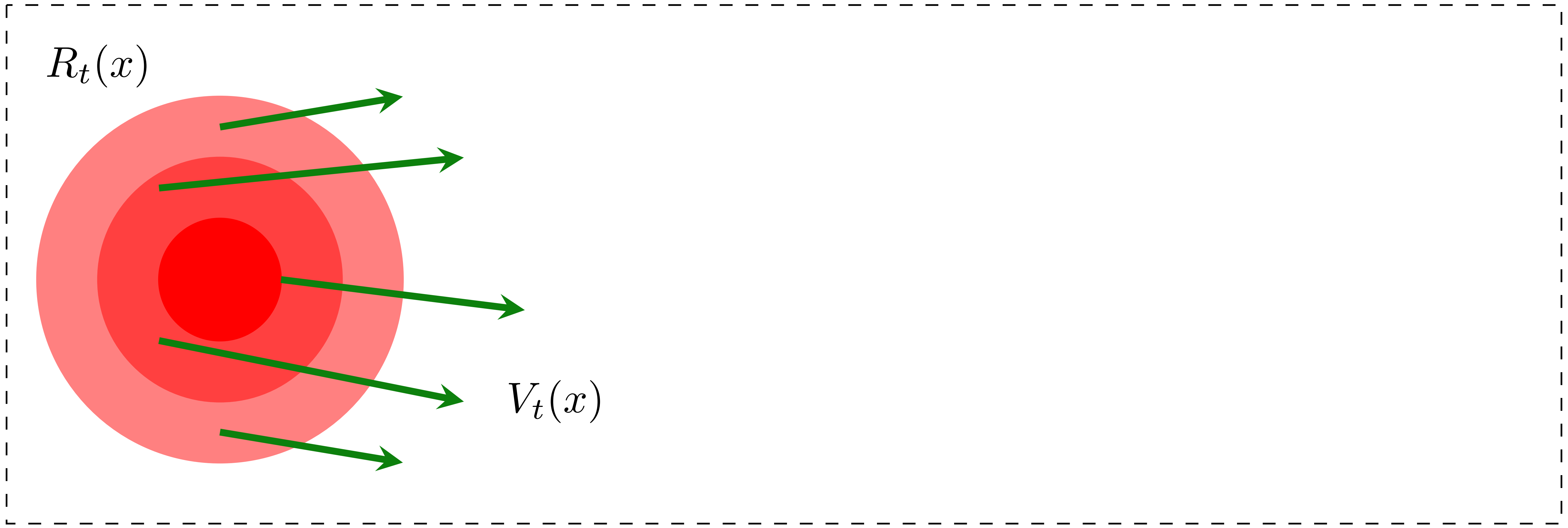
- # denotes **measure pushforward**
- Minimizer $\bar{M}_{R_t \to D_t}$ is **optimal assignment map**
- Minimum $W_2^2(R_t, D_t)$ is **Wasserstein distance**



Problem Formulation: Dynamic Model

- Tracking → want resources close to demand
- Control resource through **velocity field** V

Dynamics (Transport Equation): $\partial_t R_t(x) = -\nabla \cdot \big(R_t(x)\, V_t(x)\big)$



Motion Cost: $\|V_t\|^2_{L^2(R_t)} := \int_\Omega \|V_t(x)\|_2^2\, R_t(x)\, dx$

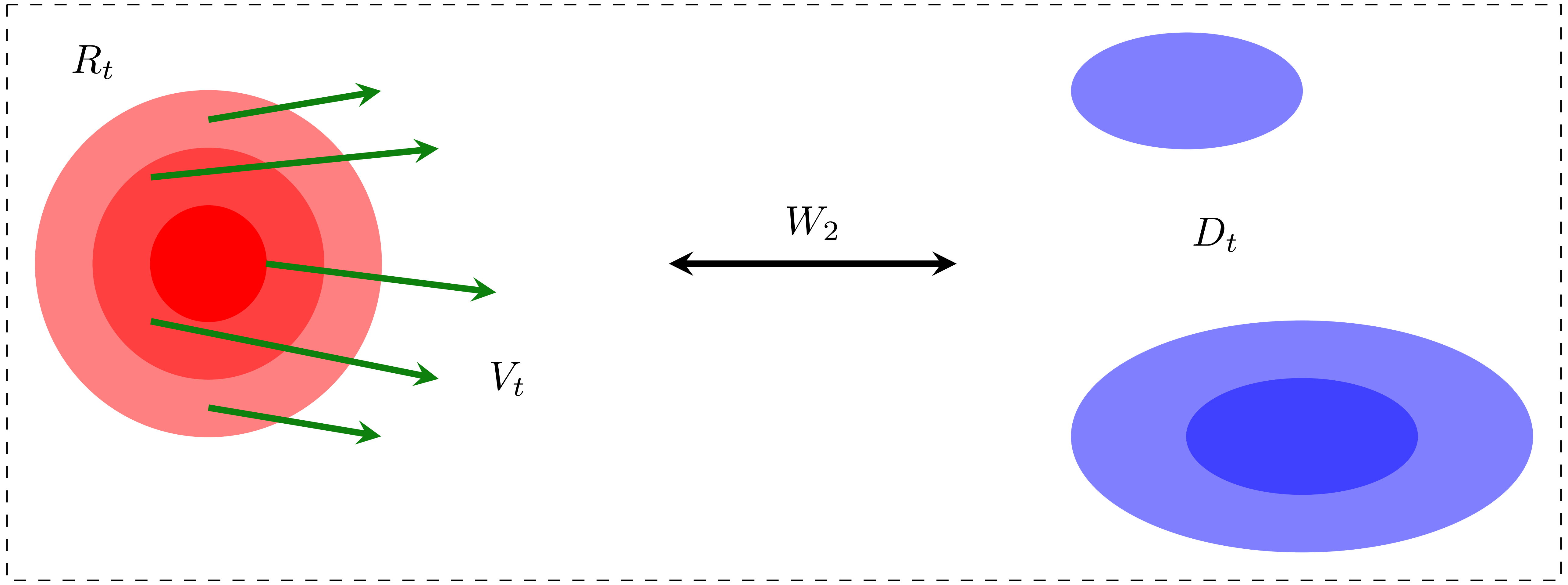
Formal Problem Statement

Given an initial resource distribution R_0 and demand trajectory D, solve

$$\inf_{R,V} \int_0^T \underbrace{W_2^2(R_t, D_t)}_{\text{Assignment Cost}} + \alpha \underbrace{\|V_t\|^2_{L^2(R_t)}}_{\text{Motion Cost}} \, dt \qquad \text{s.t.} \qquad \underbrace{\partial_t R_t = -\nabla \cdot (R_t V_t)}_{\text{Dynamic Constraint}}$$

- Intuitively, "R should track D efficiently"
- Trade-off parameter α controls relative importance of costs

Minimize cost over (R, V)



Structural Features of Solution

Necessary Conditions for Optimality:

$$\partial_t R_t = -\nabla \cdot (R_t \nabla \Lambda_t) \qquad R_0 = R_0$$

$$\partial_t \Lambda_t = -\frac{1}{2}\|\nabla \Lambda_t\|_2^2 + \frac{1}{2\alpha}\frac{\delta}{\delta R_t} W_2^2(R_t, D_t) \qquad \Lambda_T = 0$$

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Optimal velocity field is irrotational!

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$$\begin{aligned} R_0 &= R_0 \\ \Lambda_T &= 0 \end{aligned}$$

Nonlinear two-point boundary value PDE

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Demand trajectory enters through forcing term

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Requires solving an optimization problem

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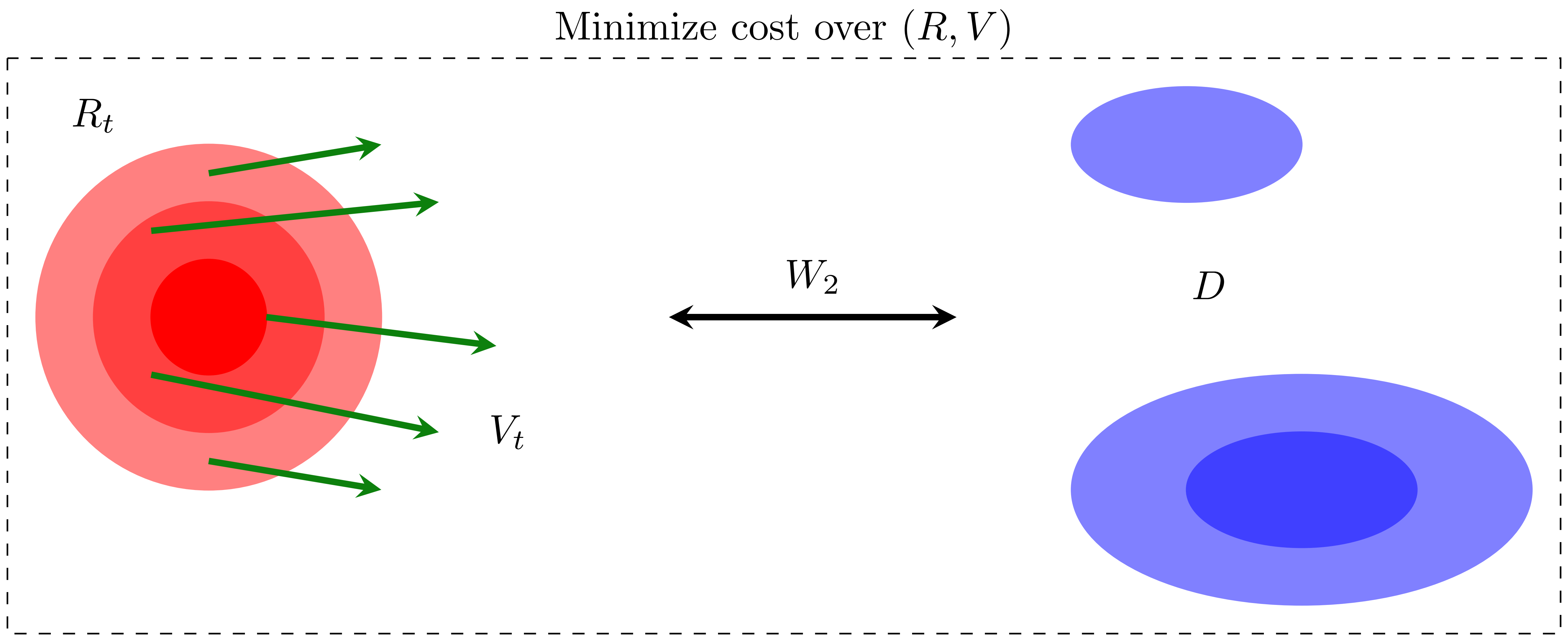
- Optimal solutions are **noncausal:** need to know D ahead of time
- Computational nightmare
- How to approach this?

Simple Case: Regulation Problem

- Consider same problem with D constant in time
- (In this case, know future trajectory of D!)

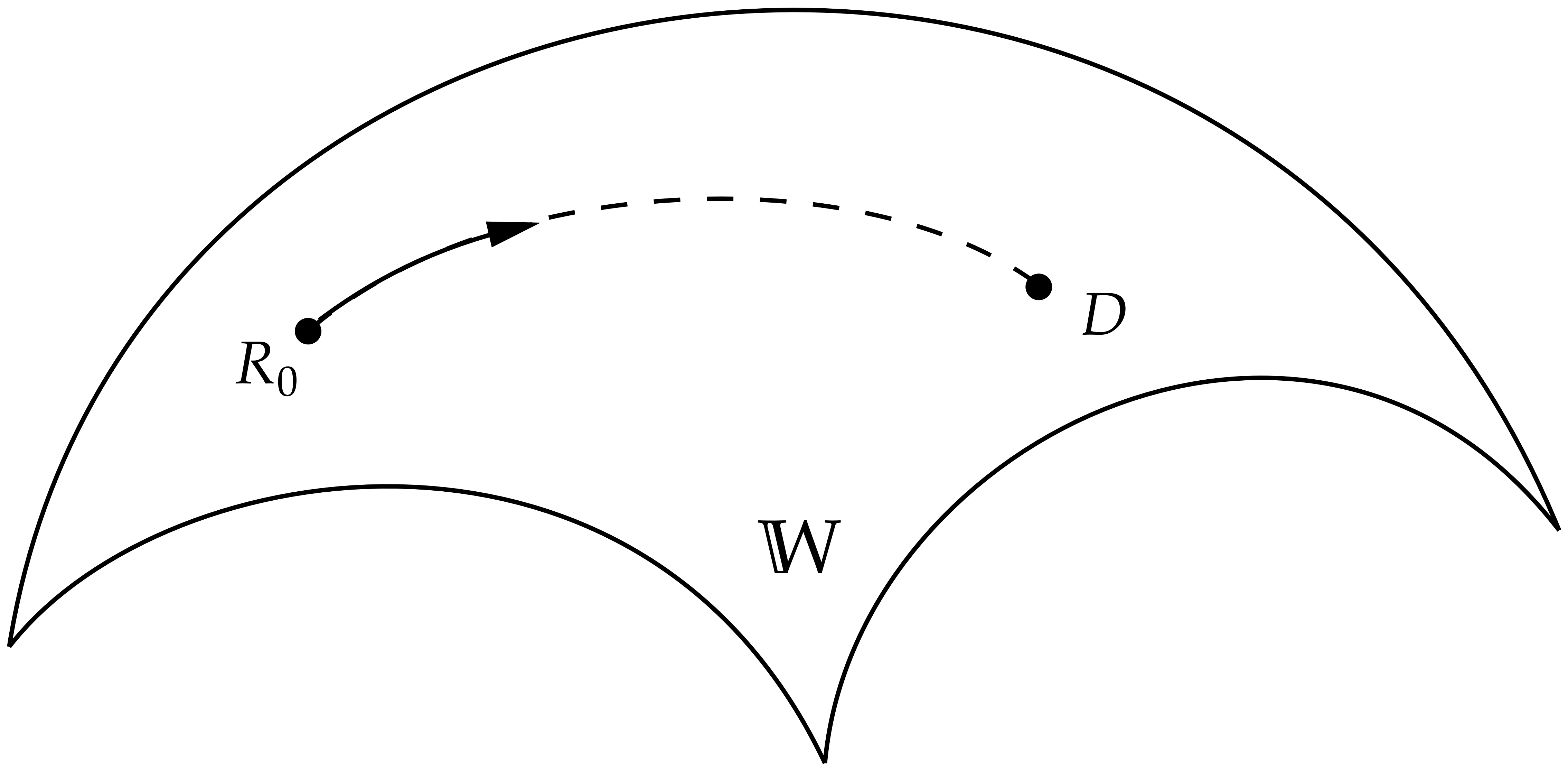
$$\inf_{R,V} \int_0^T W_2^2(R_t, D) \; + \; \alpha \, \|V_t\|^2_{L^2(R_t)} \, dt \qquad \text{s.t.} \qquad \partial_t R_t = -\nabla \cdot (R_t V_t)$$



Fact: Wasserstein Space = Riemannian Manifold

- Leverage geometric structure to solve problem
- Can show that R moves along Wasserstein geodesic towards D:

$$R_t \;=\; \left[(1-\sigma(t))\; I \;+\; \sigma(t)\; \bar{M}_{R_0 \to D}\right]_{\#} R_0$$



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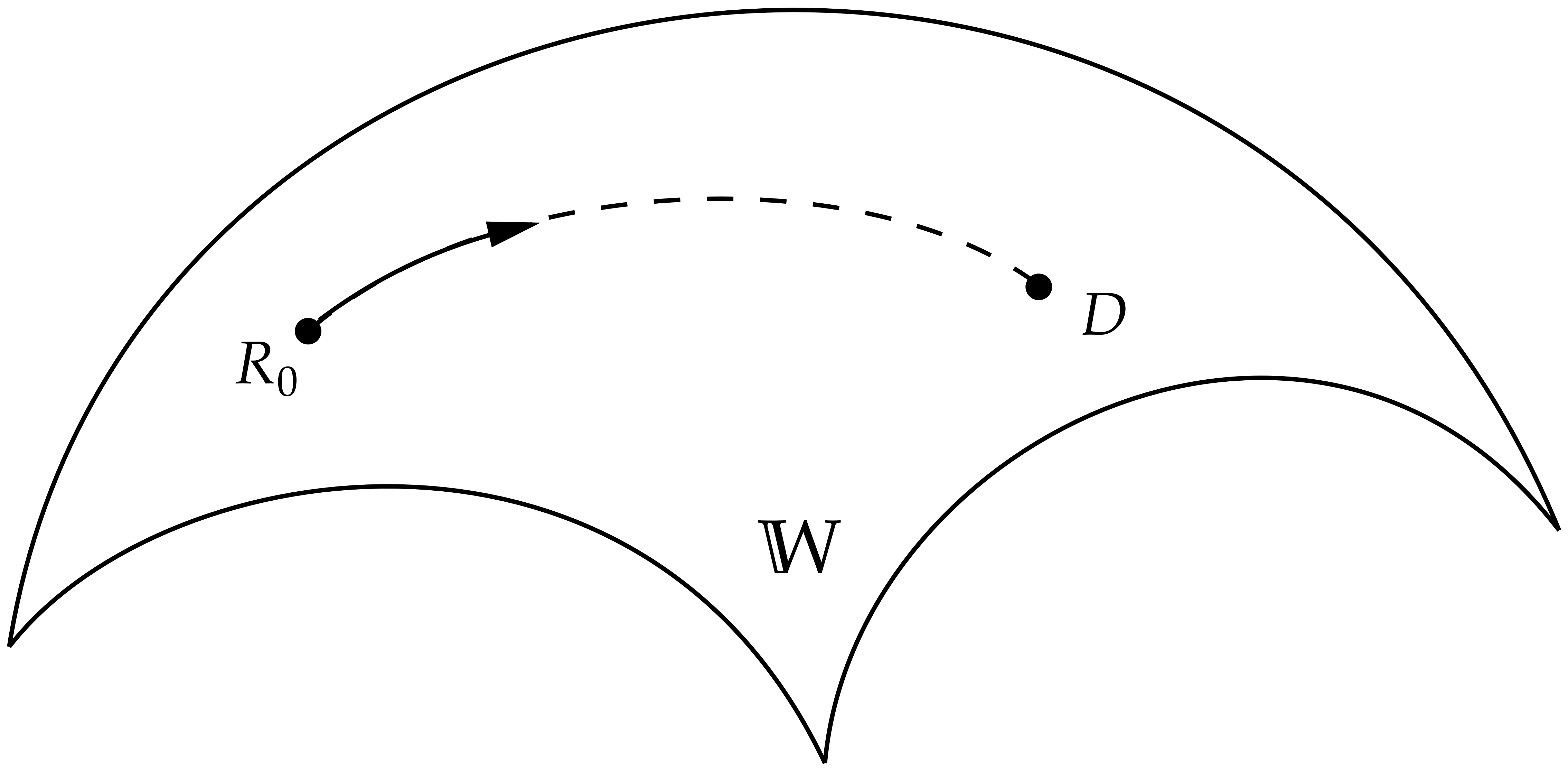
Interpolation between identity and assignment map



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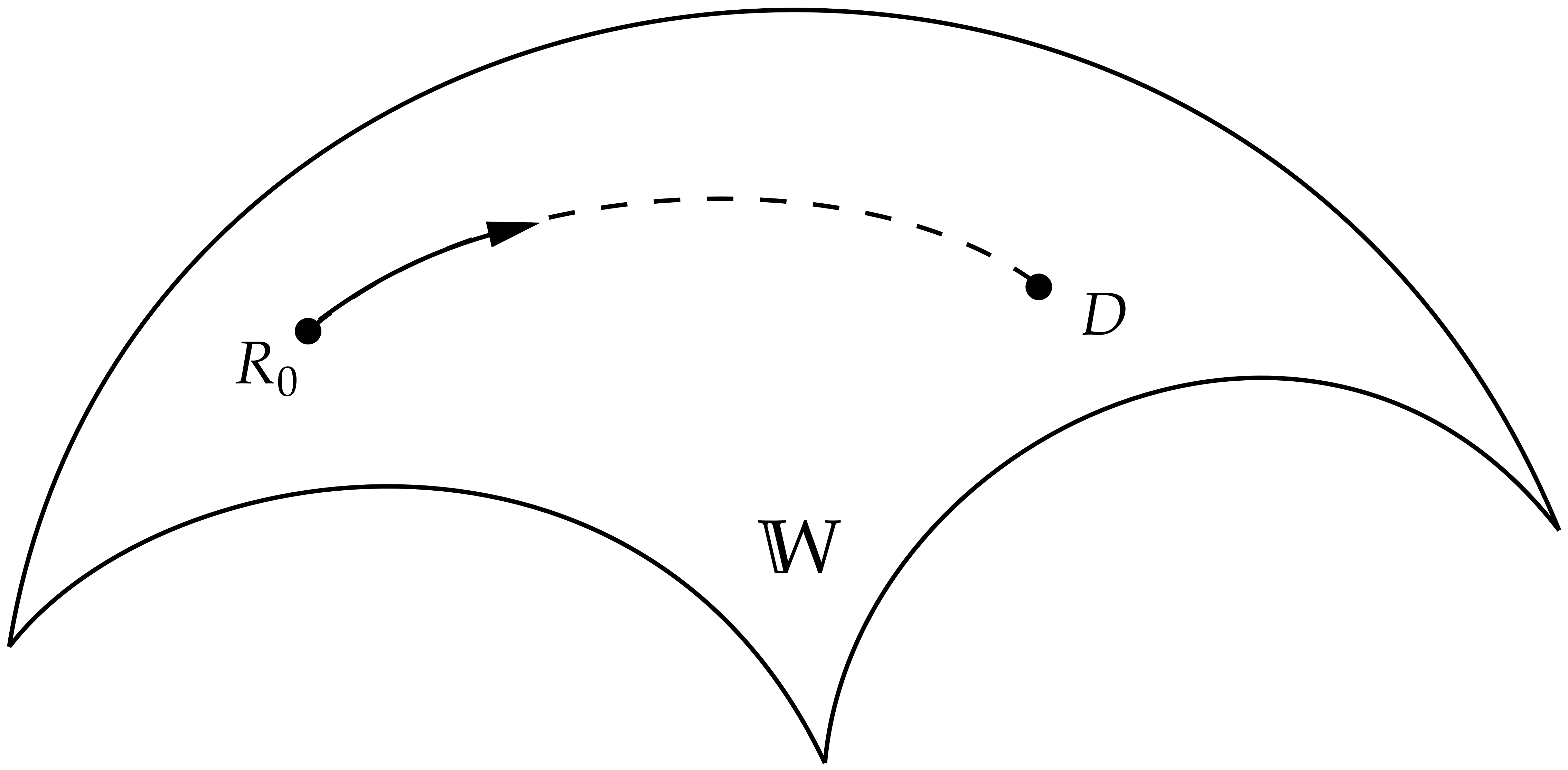
Assignment map comes from OT



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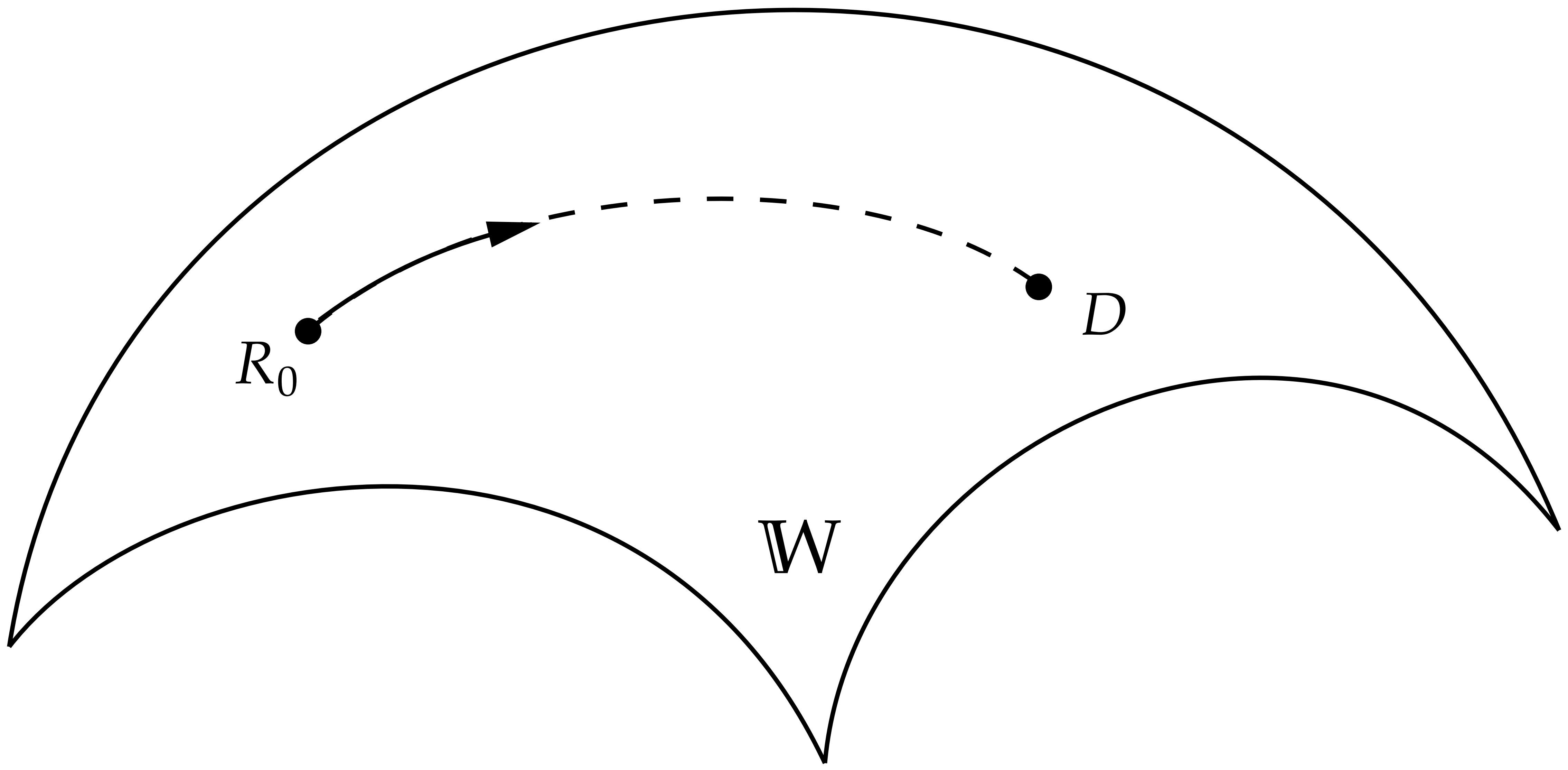
Time schedule σ comes from OC problem



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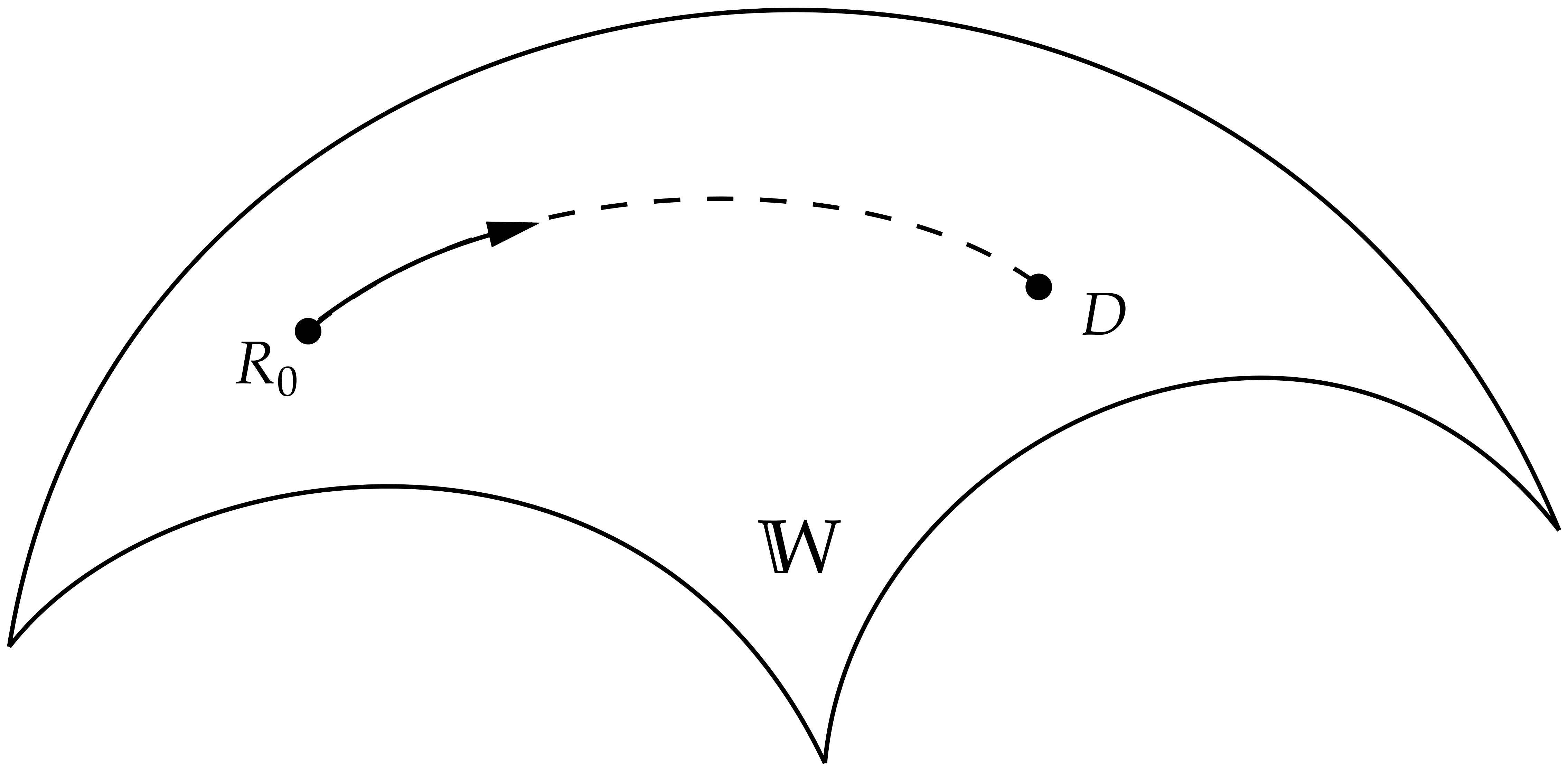

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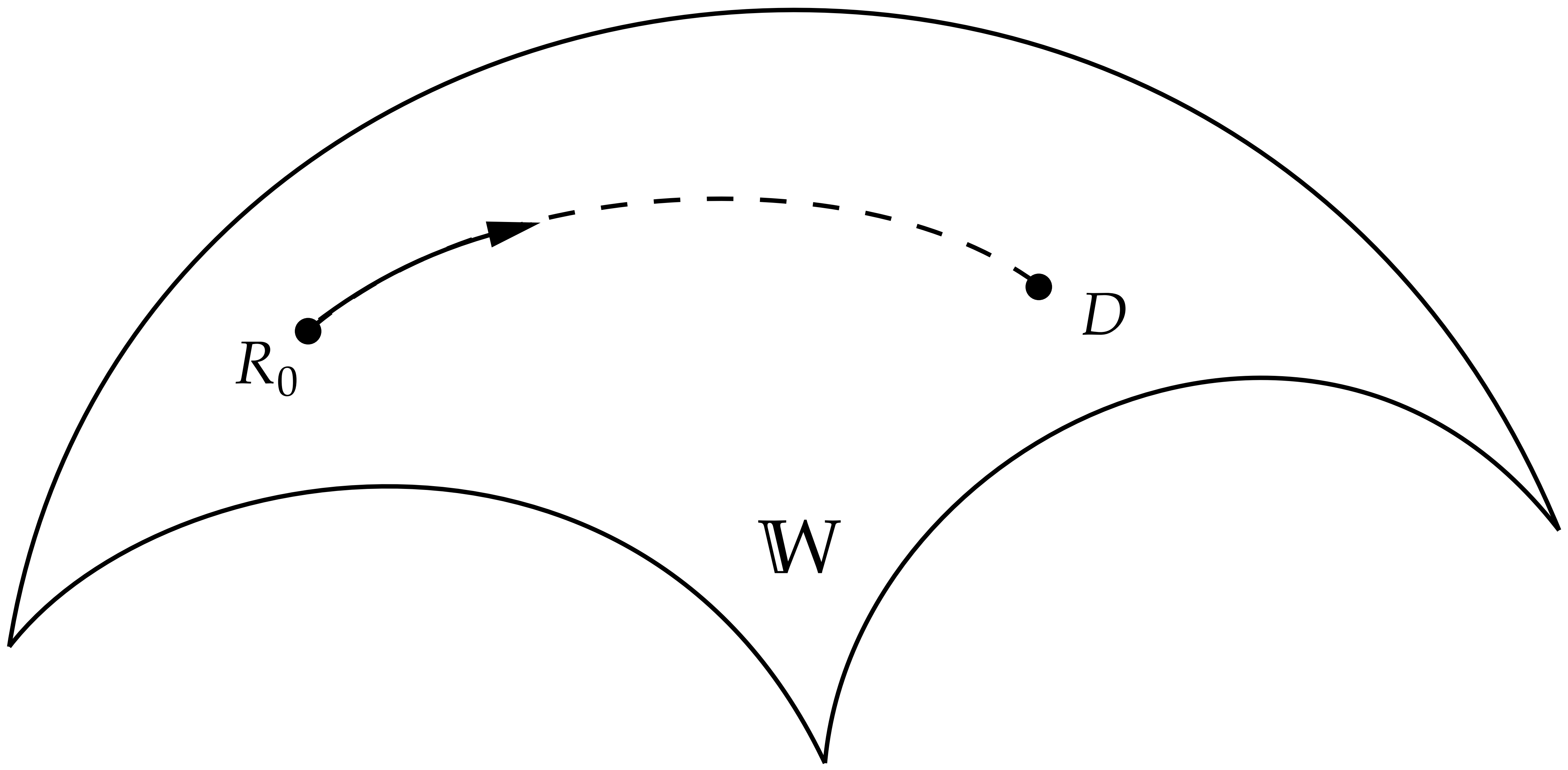

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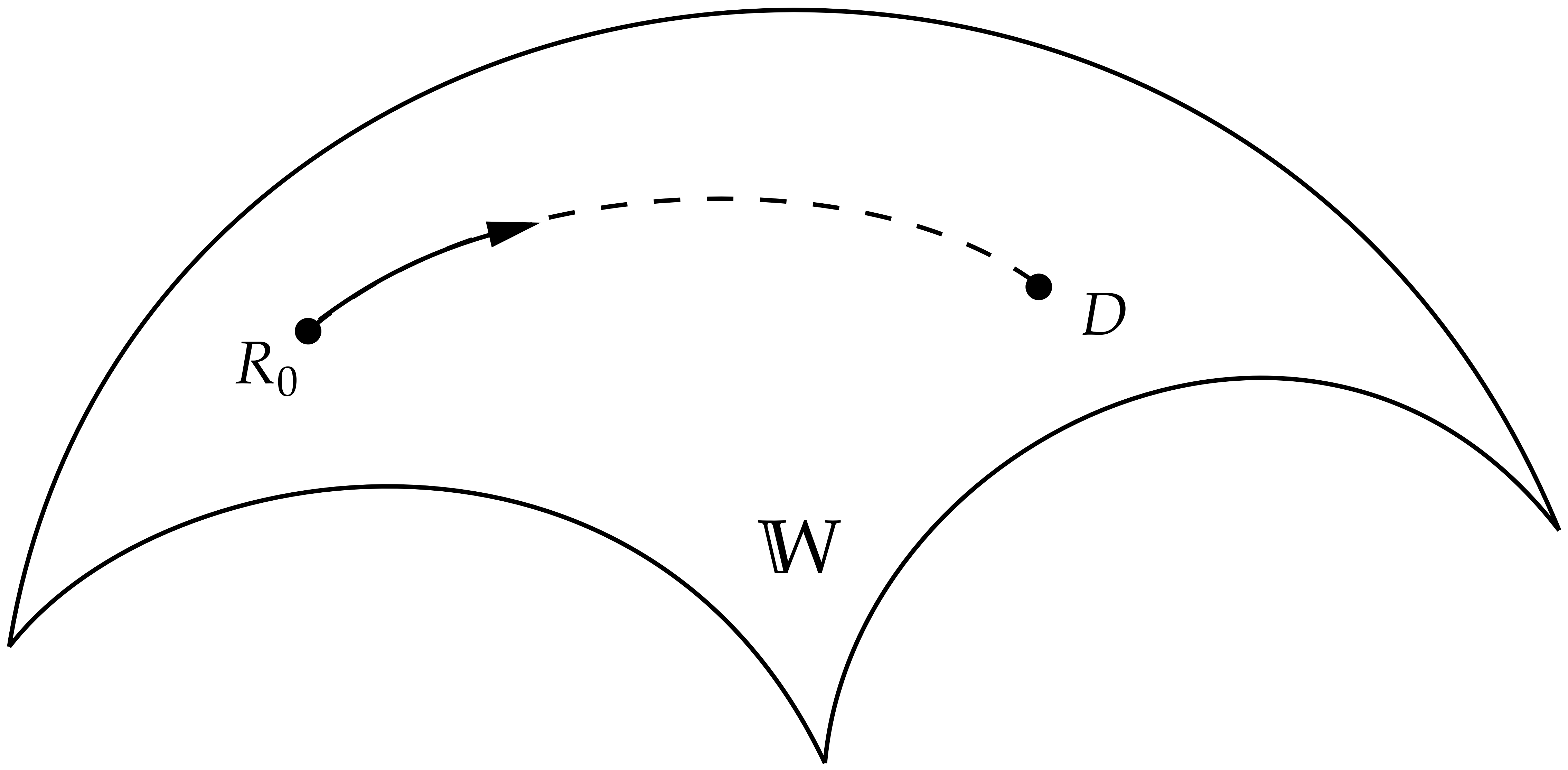
Error vector



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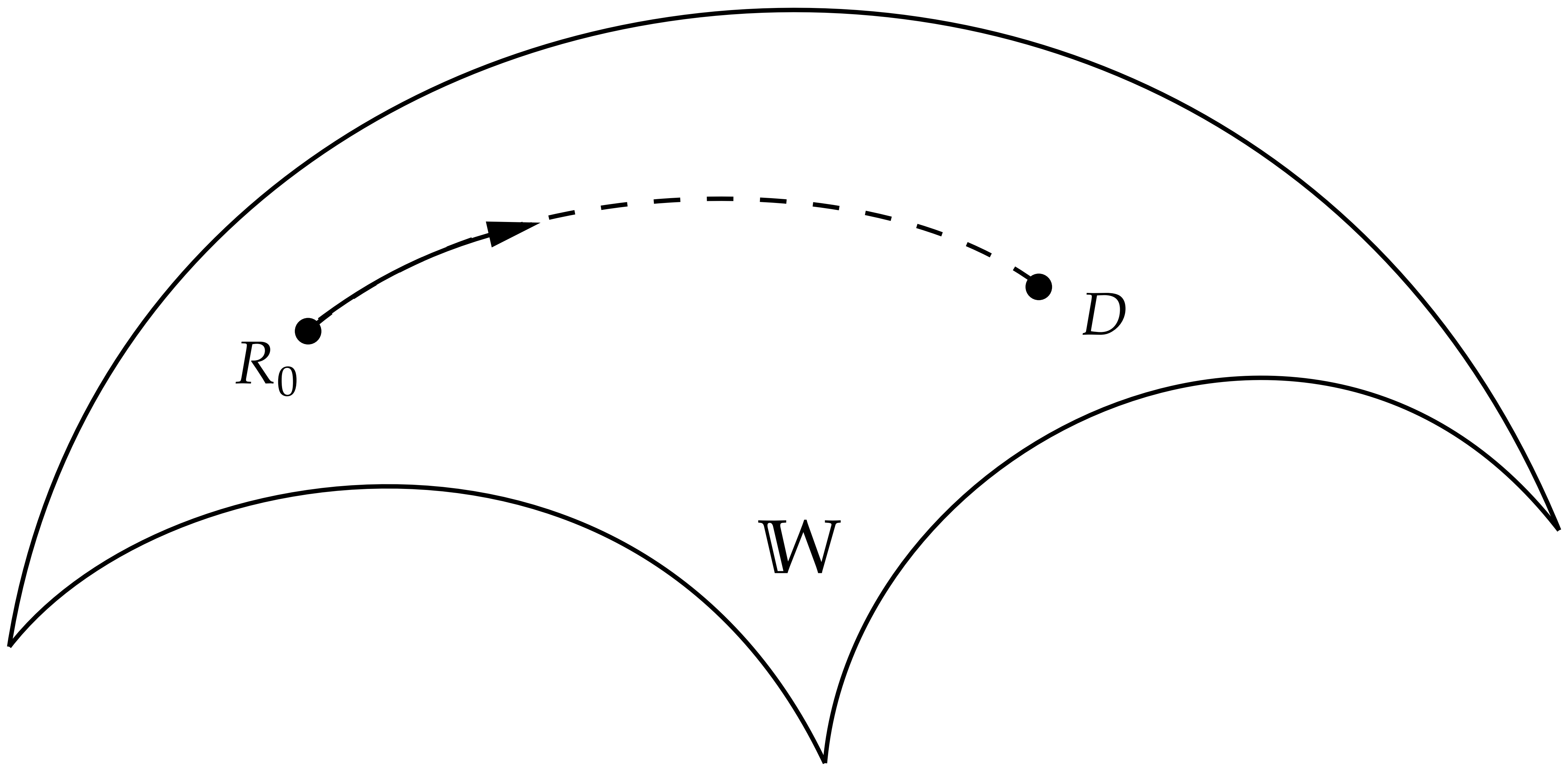
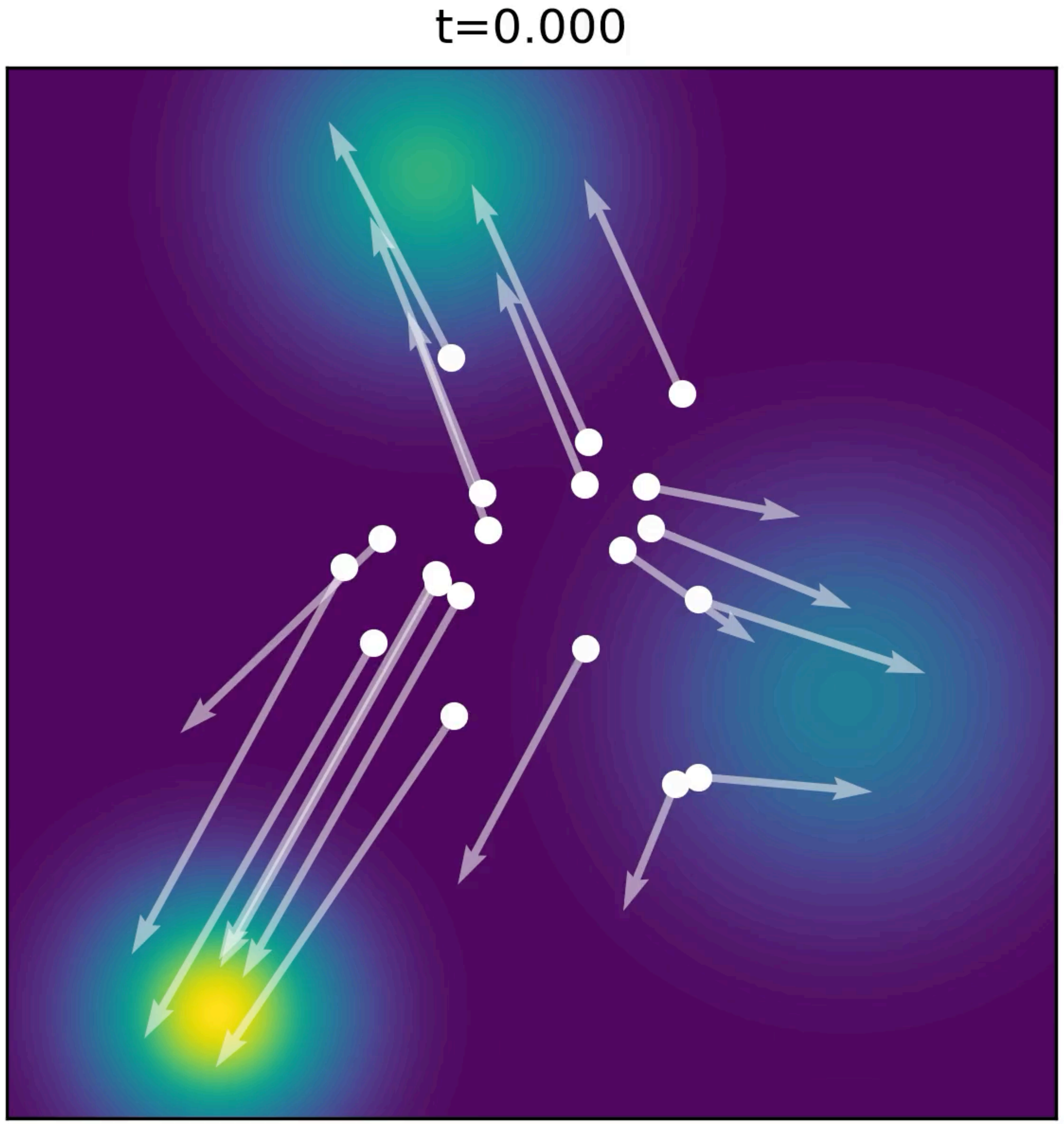
Rate of traversal



Regulation Problem: Simulation



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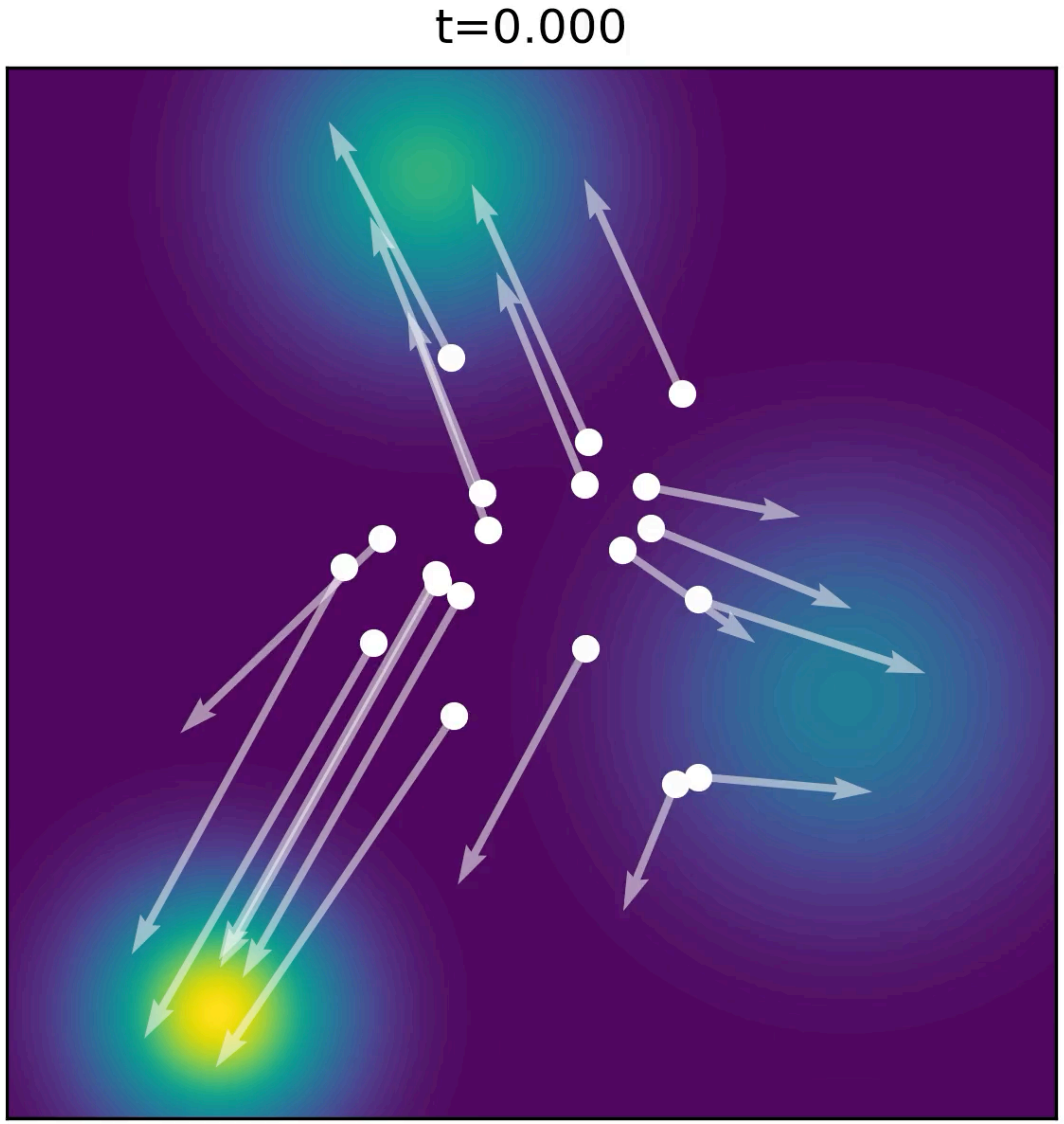

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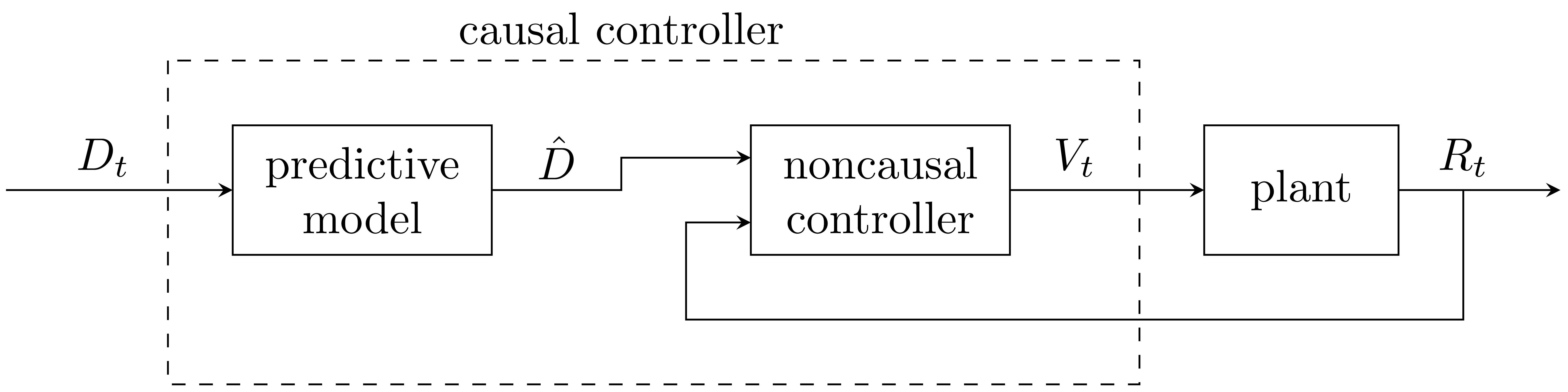

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- (Also solves **Problem #3:** computational cost)

MPC Algorithm for Time-Varying Demands

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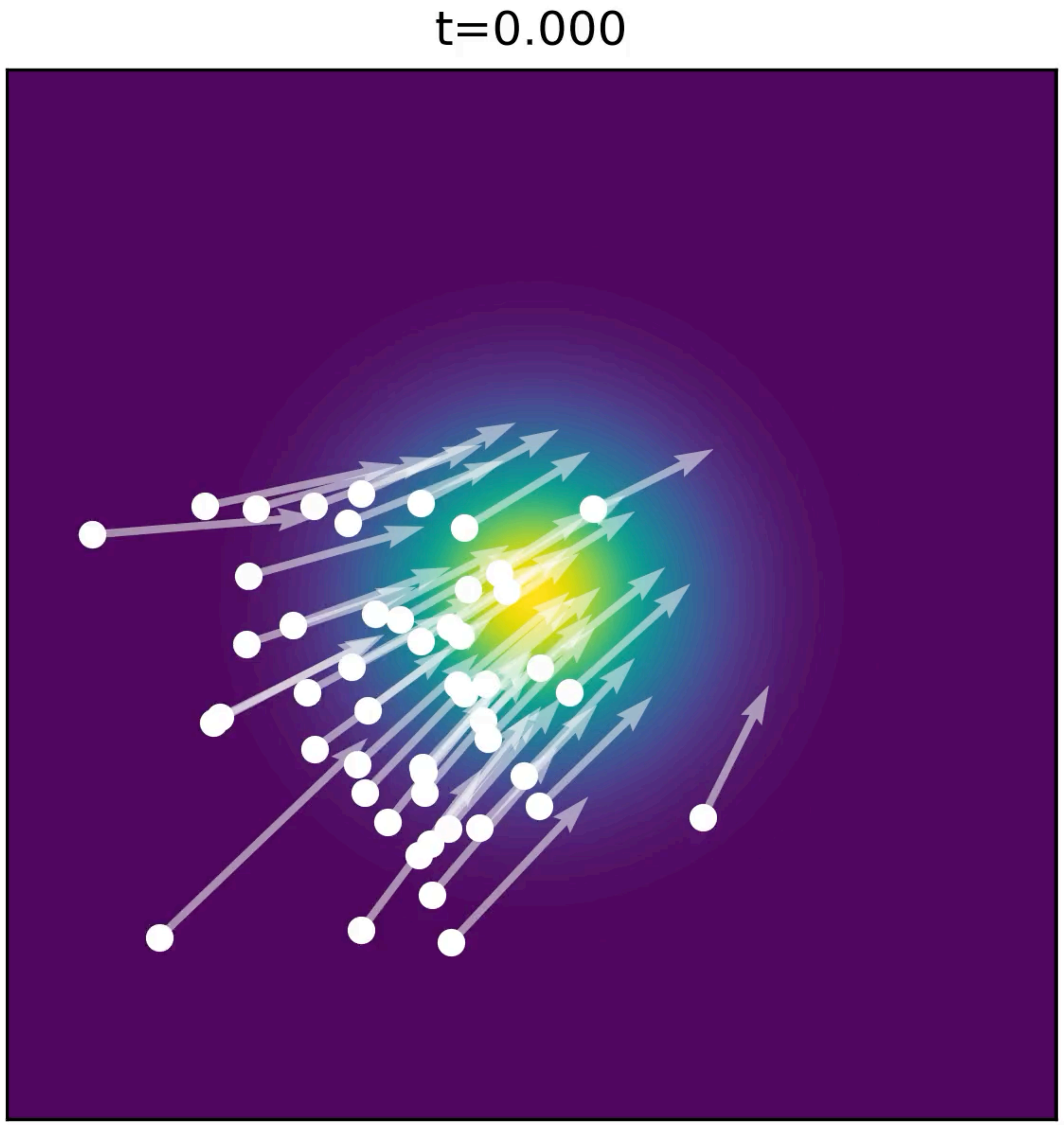
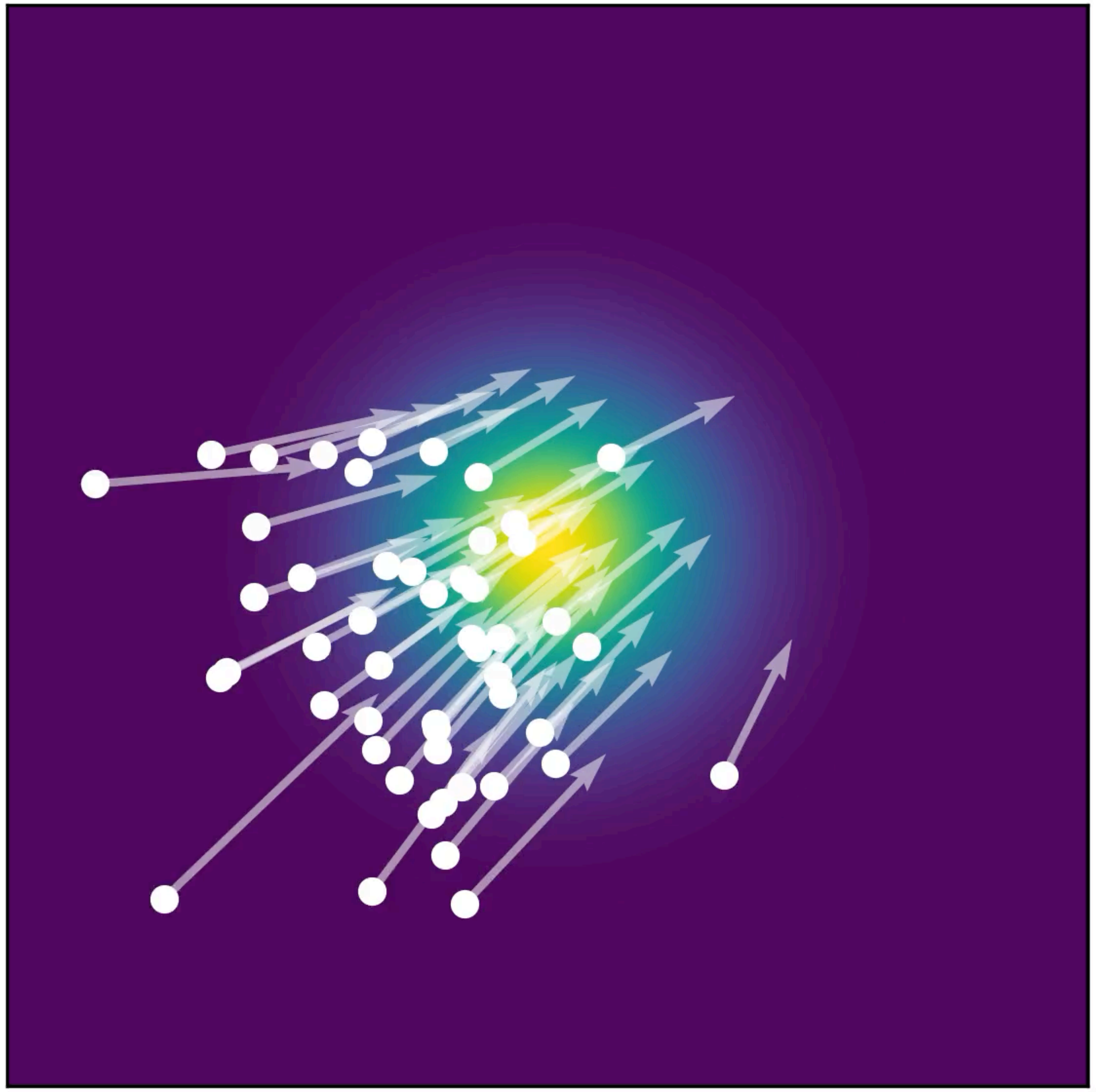
- Heuristic, not optimal in general
- But, can be applied in real-time

MPC Algorithm: Simulations



MPC Algorithm: Simulations

t=0.000



Conclusion

Takeaways:

- Simplified models can provide insight and design heuristics
- Leveraging geometric structure can be powerful

Future Work:

- Solving necessary conditions
- More sophisticated demand models
- Investigating resulting controllers

Thanks to My Collaborators

Bassam Bamieh

Jared Jonas

Thanks for watching!

Questions?