

Mean-Field Oscillator Ising Machines

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Max Emerick, Arvind R. Venkatakrishnan, Bassam Bamieh, Francesco Bullo
(University of California, Santa Barbara)

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Oscillator Ising Machines (OIMs)

Discrete Problem:

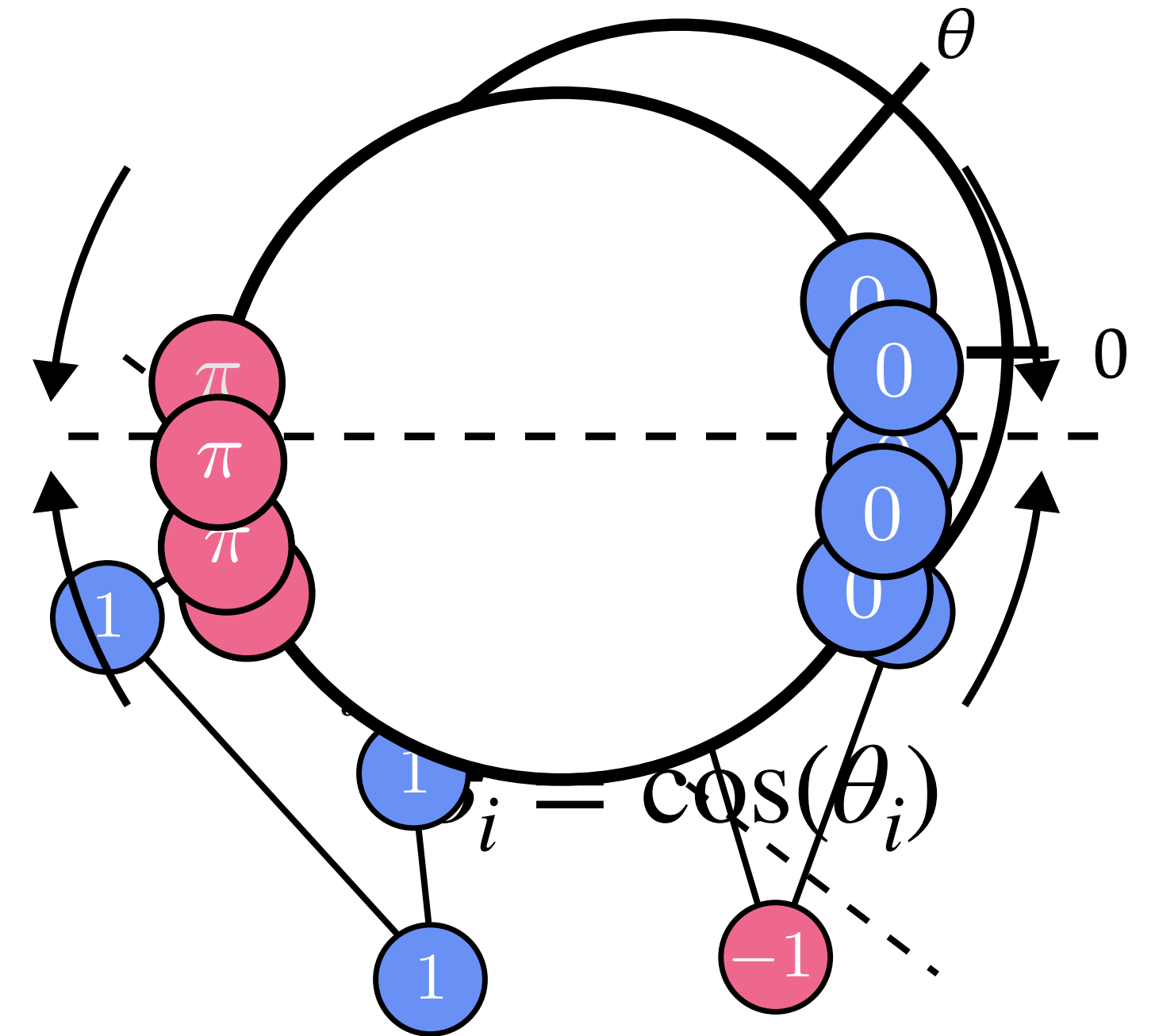
$$\min_{\sigma \in \{\pm 1\}^n} \frac{1}{2} \sum_{i,j} A_{ij} \sigma_i \sigma_j$$

Continuous Problem:

$$\min_{\theta \in \mathbb{T}^n} \frac{1}{2} \sum_{i,j} A_{ij} \cos(\theta_i - \theta_j) - \frac{K_s}{2} \sum_i \cos(2\theta_i)$$

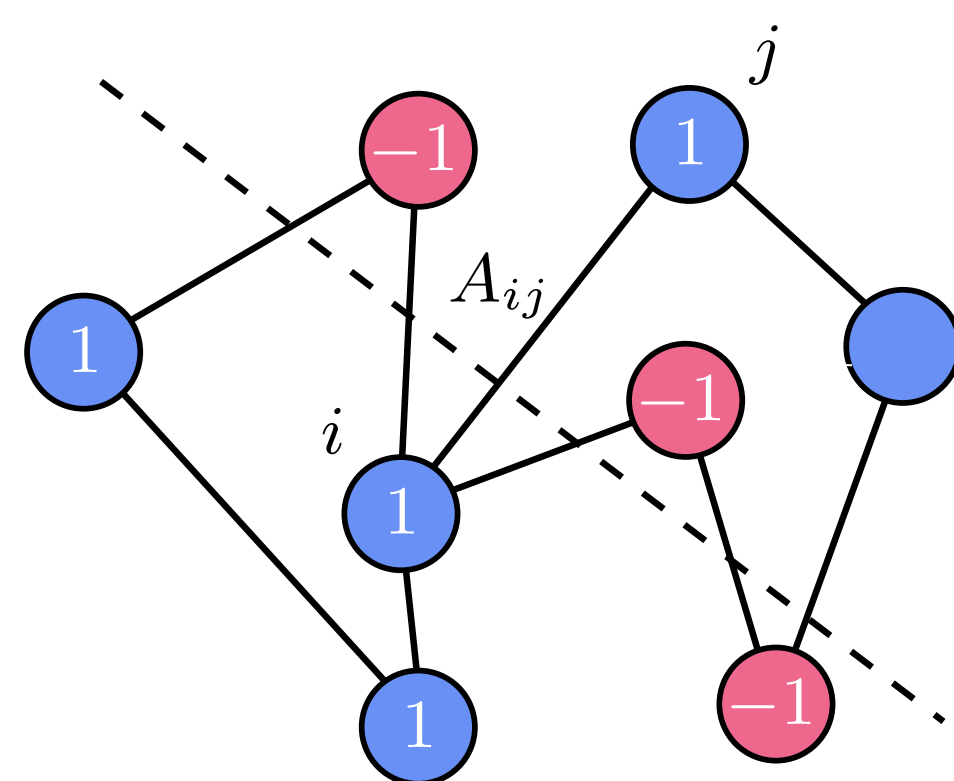
Gradient Descent:

$$\dot{\theta}_i = \sum_j A_{ij} \sin(\theta_i - \theta_j) - K_s \sin(2\theta_i)$$



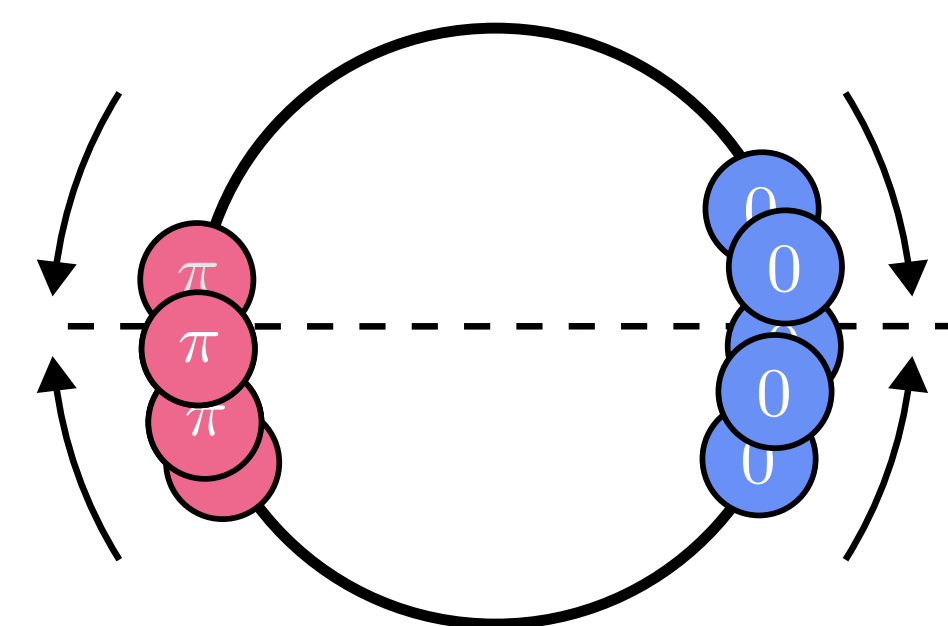
Oscillator Ising Machines (OIMs)

Max-Cut



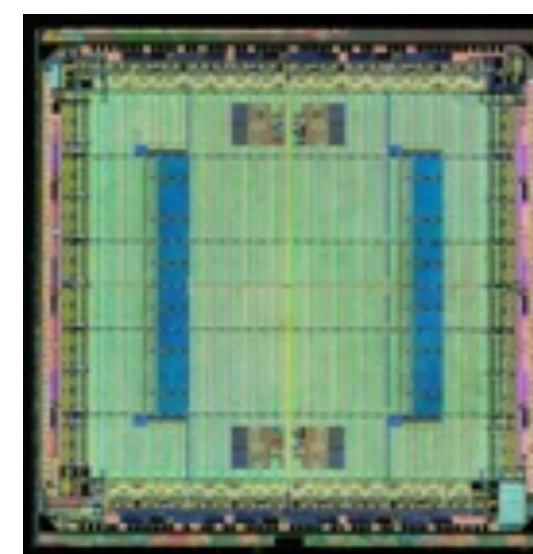
$$\min_{\sigma \in \Sigma^n} \frac{1}{2} \sum_{i,j} A_{ij} \sigma_i \sigma_j$$

Oscillator Ising Machines



$$\dot{\theta}_i = \sum_j A_{ij} \sin(\theta_i - \theta_j) - K_s \sin(2\theta_i)$$

Illustrative implementation using FPGA



Main Questions

- How do these OIMs behave? Can we give guarantees?
- Can we always recover feasible solutions? Does $\theta \rightarrow \{0, \pi\}$?
- How often do we get good solutions?
- How do we design for performance? What should K_s be?

Outline of Talk

- Investigate mean field ($n \rightarrow \infty$) limits of these OIMs
- Two different kinds:
 - Lagrangian (always valid)
 - Eulerian (reduced order through symmetry)
- Both limits have gradient flow structure
- Develop classification of fixed points in symmetric case
- Show numerical evidence that classification predicts behavior in certain large random networks

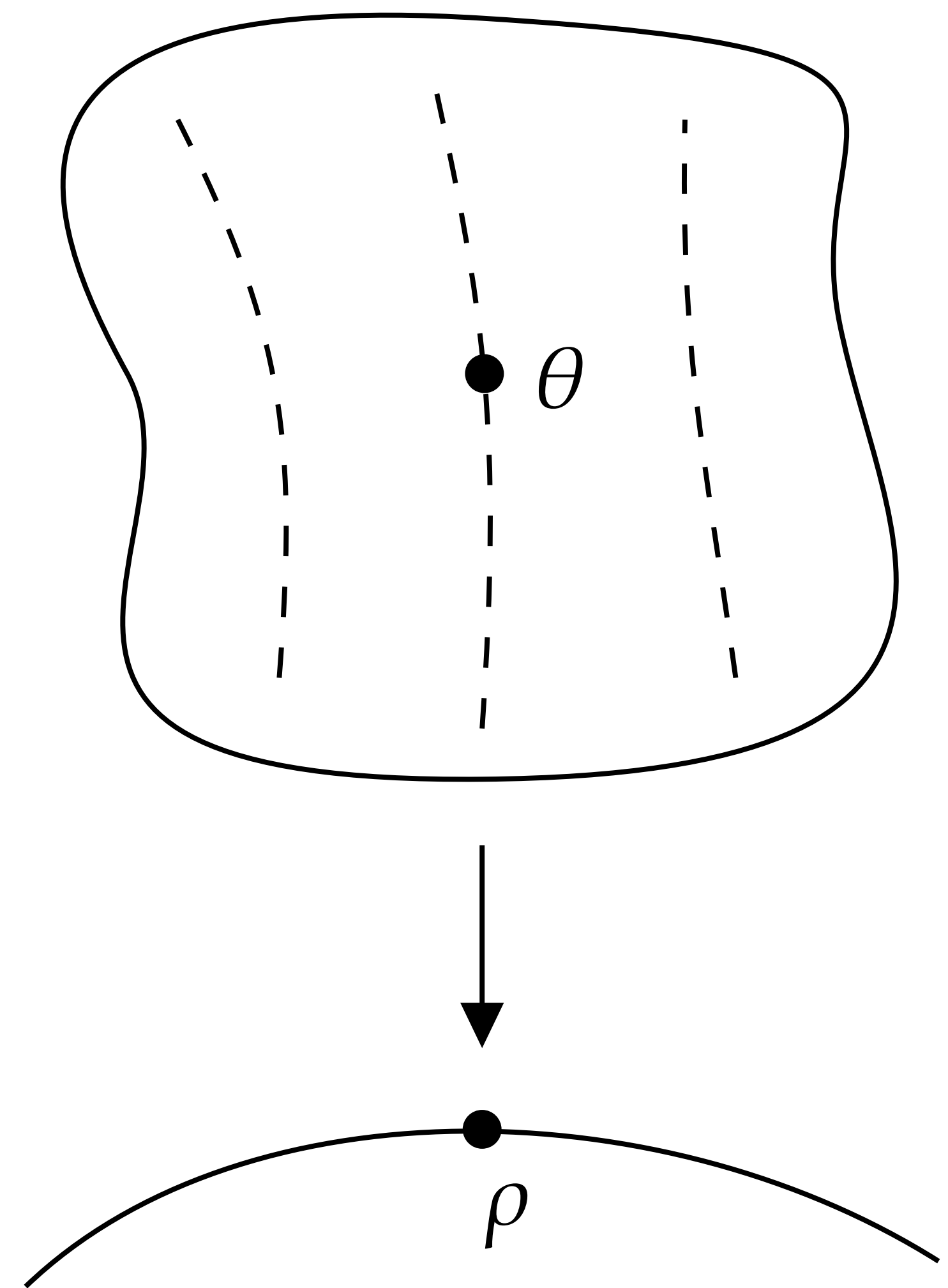
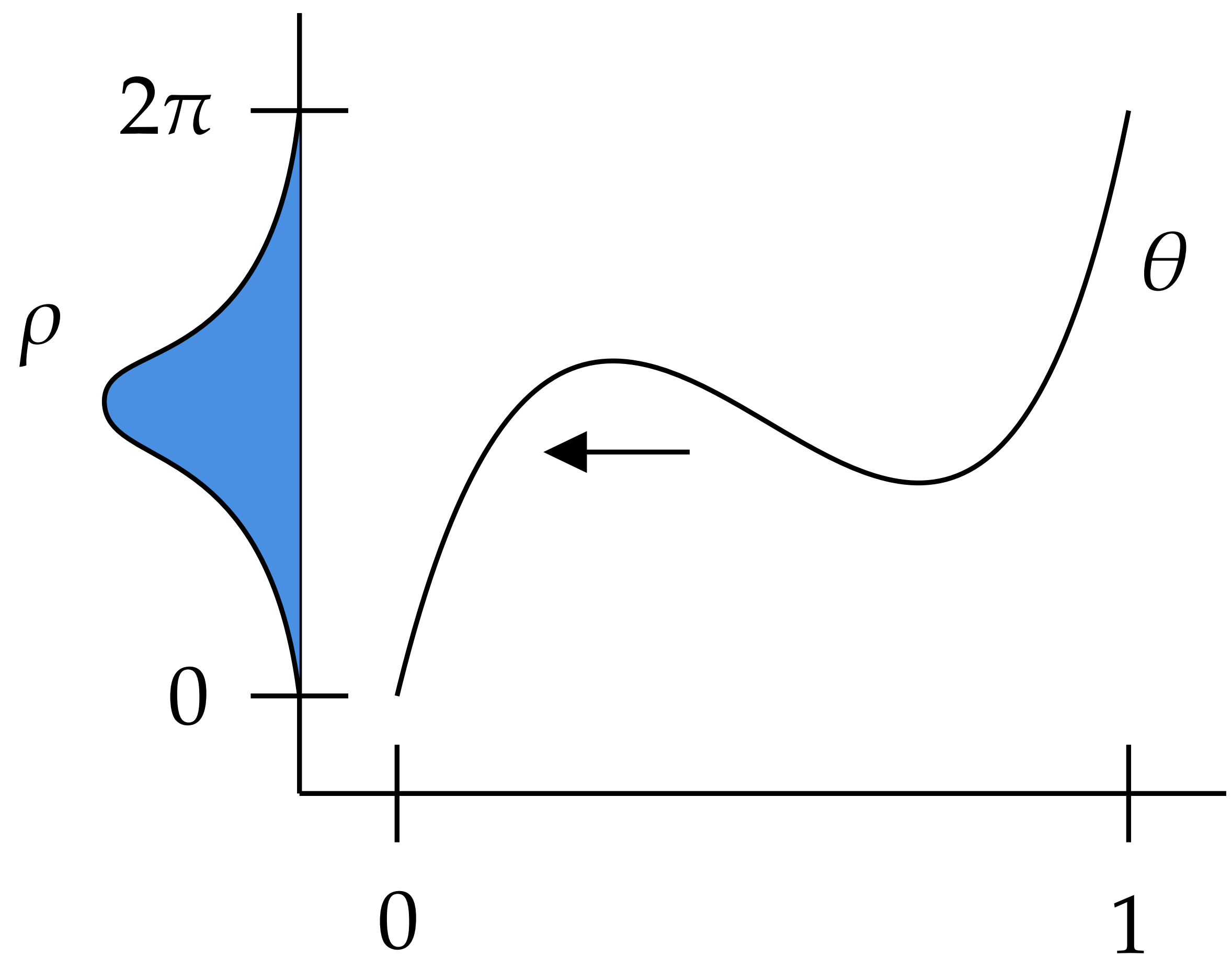
Lagrangian Mean-Field Limit

- Index set $\mathfrak{n} = \{1, \dots, n\}$ converges to $I = [0,1]$
- Adjacency matrix $A : \mathfrak{n} \times \mathfrak{n} \rightarrow \mathbb{R}$ converges to graphon $K : I \times I \rightarrow \mathbb{R}$
- State $\theta : \mathfrak{n} \rightarrow \mathbb{T}$ converges to “phase function” $\theta : I \rightarrow \mathbb{T}$
- Lagrangian energy and dynamics derived formally by passing to continuum:

$$E(\theta) = \frac{1}{2} \iint K(x, y) \cos(\theta(x) - \theta(y)) \, dx \, dy - \frac{K_s}{2} \int \cos(2\theta(x)) \, dx$$

$$\partial_t \theta(x) = \int K(x, y) \sin(\theta(x) - \theta(y)) \, dy - K_s \sin(2\theta(x))$$

Eulerian-Lagrangian Correspondence



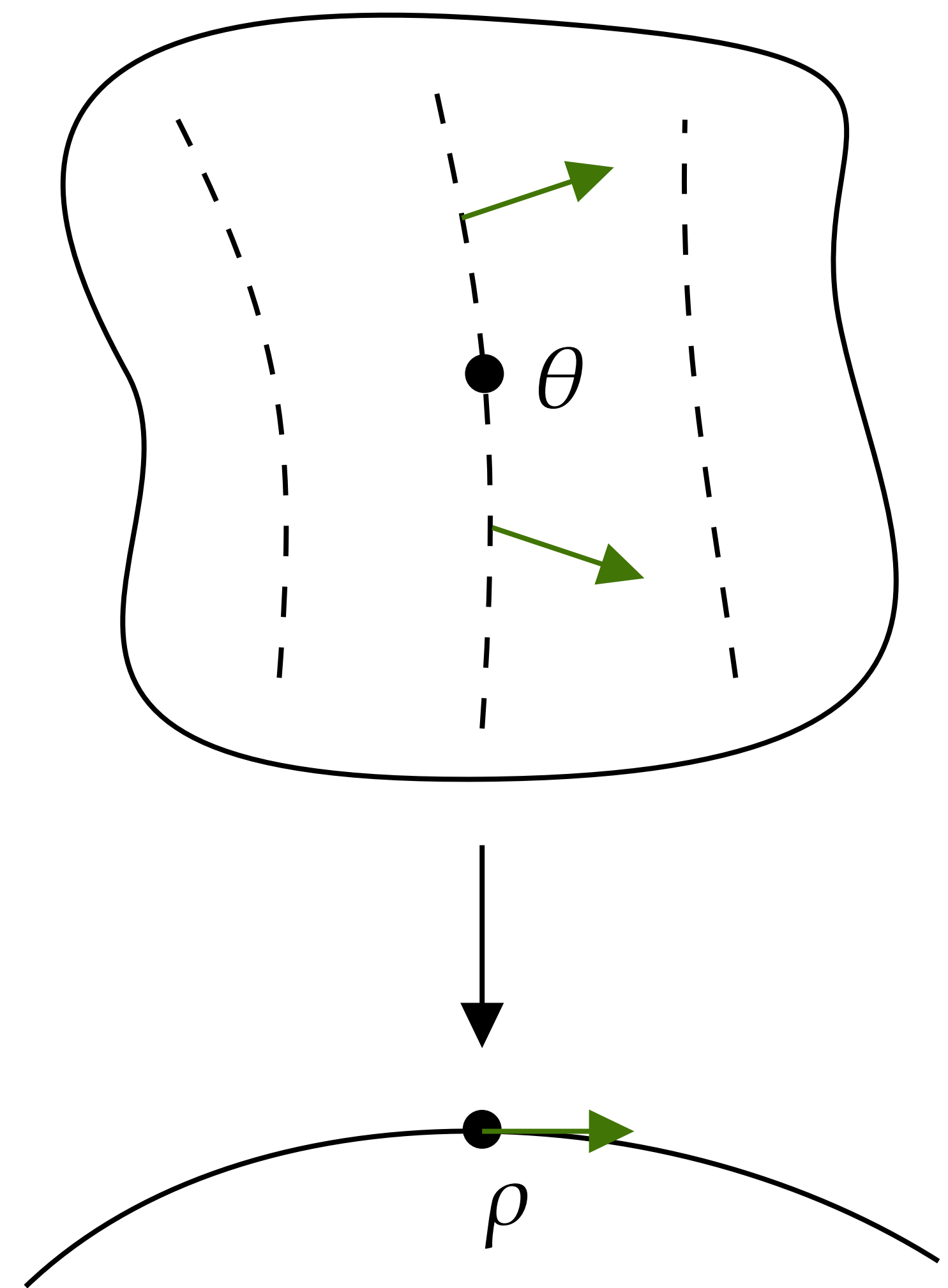
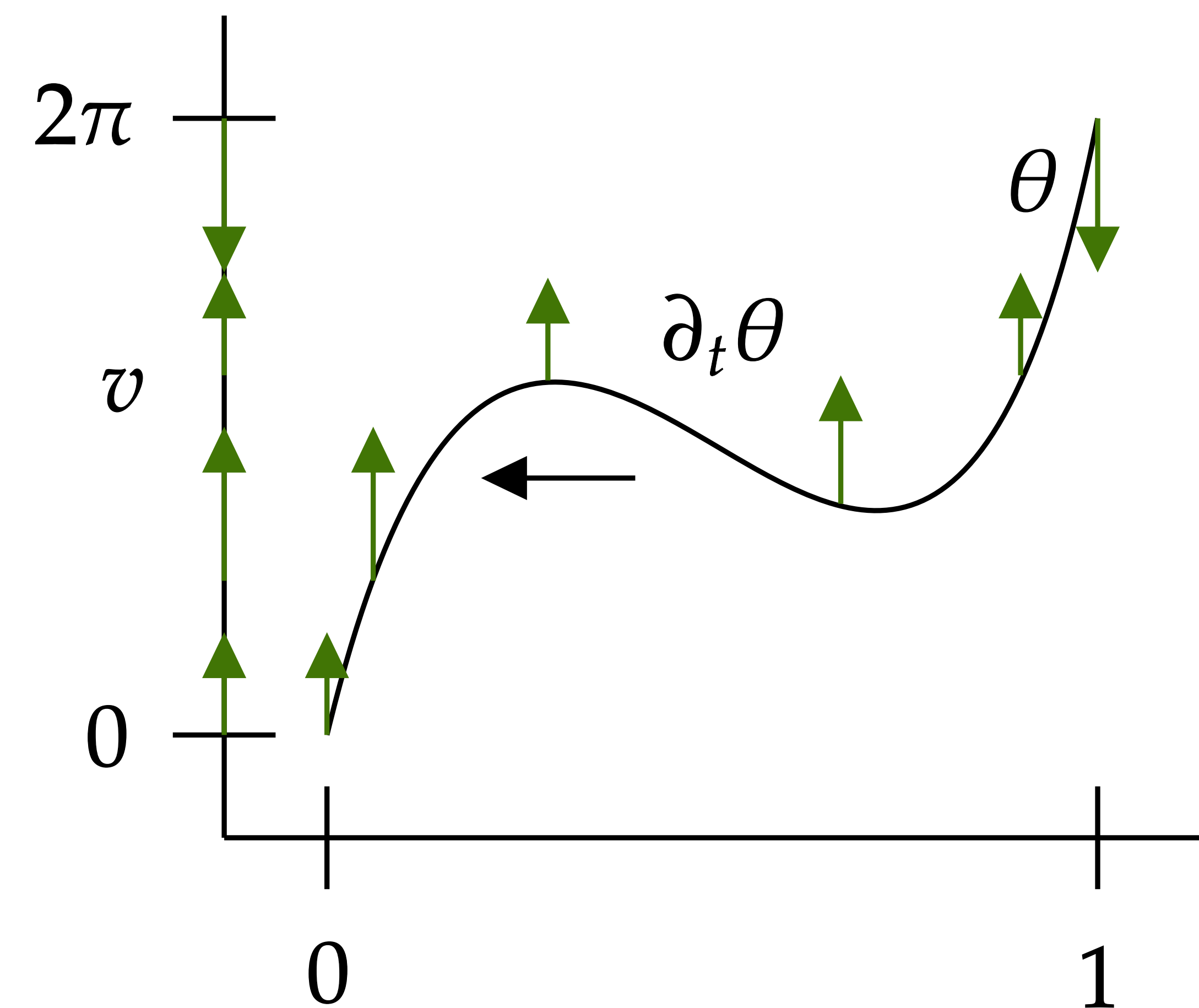
Eulerian Mean-Field Limit

- Phase density ρ always defined but does not always form a state
- ρ forms a state $\Leftrightarrow$ dynamics $\partial_t \theta$ can be pushed forward to $\partial_t \rho$
 - $\Leftrightarrow$ dynamics are invariant on relabeling equivalence classes
 - $\Leftrightarrow K(x, y) \equiv K = \text{constant}$
- In this case, Eulerian energy and dynamics given by:

$$E(\rho) = \frac{K}{2} \iint \cos(\theta - \phi) \rho(\theta) \rho(\phi) d\theta d\phi - \frac{K_s}{2} \int \cos(2\theta) \rho(\theta) d\theta$$

$$\partial_t \rho(\theta) = -\partial_\theta (\rho(\theta) v(\theta)) = -\partial_\theta \left(\rho(\theta) [K (\sin \star \rho)(\theta) - K_s \sin(2\theta)] \right)$$

Eulerian Mean-Field Limit



Main Problem

Problem: Classify fixed points and stability of constant- K mean-field models:

$$\text{Lagrangian :} \quad \partial_t \boldsymbol{\theta}(x) = K \int \sin(\boldsymbol{\theta}(x) - \boldsymbol{\theta}(y)) dy - K_s \sin(2\boldsymbol{\theta}(x))$$

$$\text{Eulerian :} \quad \partial_t \boldsymbol{\rho}(\theta) = -\partial_\theta \left(\boldsymbol{\rho}(\theta) \left[K (\sin \star \boldsymbol{\rho})(\theta) - K_s \sin(2\theta) \right] \right)$$

Approach: Leverage the gradient flow structure

Background: Gradient Flows

- Classical (Euclidean) gradient flow: $\dot{x} = -\nabla E(x)$
- Can be extended to more general settings
- Gradient flows are highly structured:
 - Dissipation identity: $\dot{E}(x) = -\|\nabla E(x)\|^2$
 - Expect trajectories converge to fixed points ($\nabla E(x) = 0$)
 - Expect generic trajectories converge to local minima ($\nabla^2 E(x) > 0$)
 - ∇E and $\nabla^2 E$ admit closed-form expressions via calculus of variations

Equilibrium Conditions

- ρ is a fixed point $\Leftrightarrow \rho$ is supported on zeros of vector field

$$v_\rho(\theta) = K(\sin \star \rho)(\theta) - K_s \sin(2\theta)$$

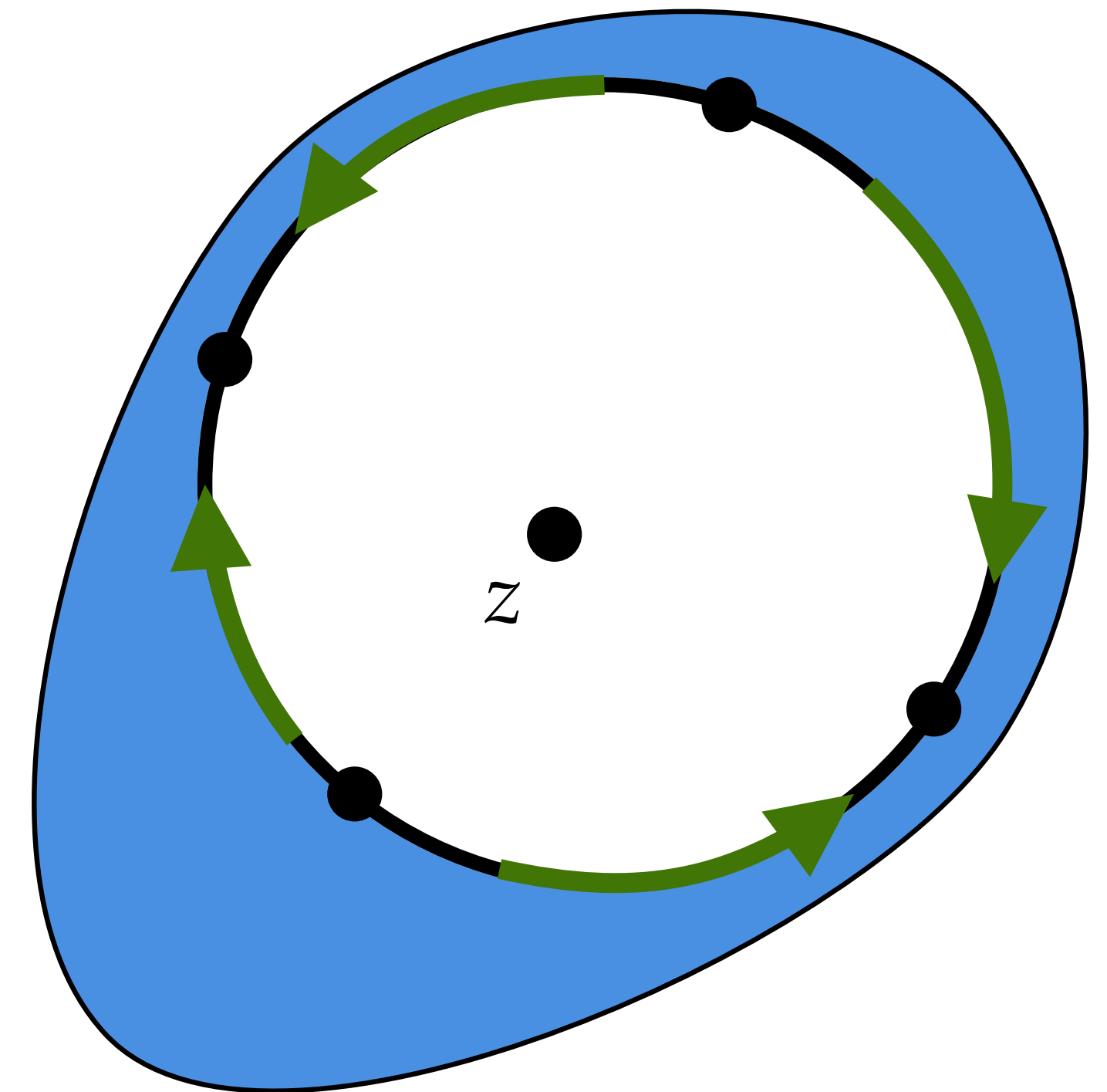
$\Rightarrow$ phases of oscillators can only cluster at zeros

- v finitely parameterized by order parameter:

$$v_z(\theta) = -K \operatorname{Im}(ze^{-i\theta}) - K_s \sin(2\theta),$$

$$z := \int e^{i\theta} \rho(\theta) d\theta$$

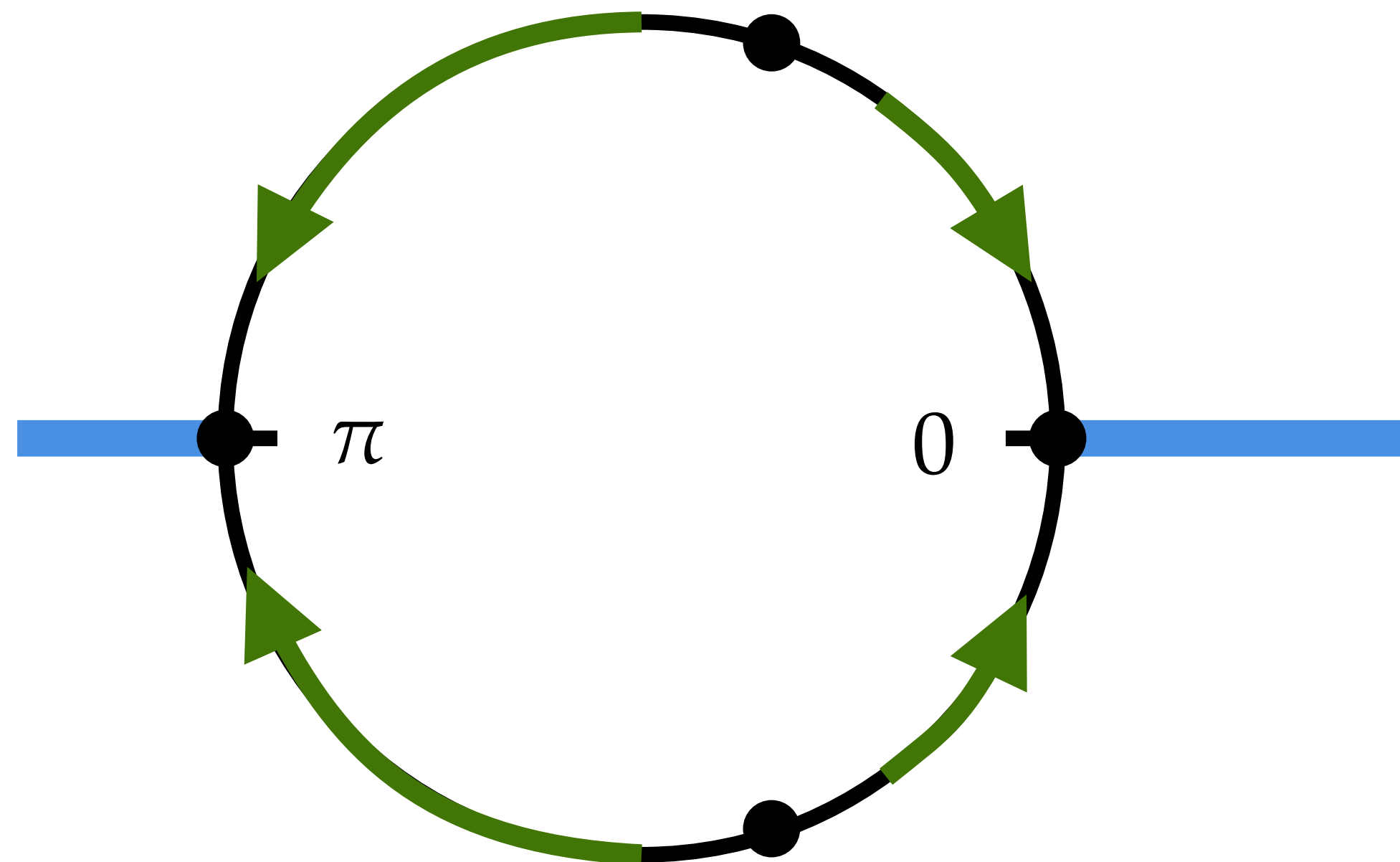
$\Rightarrow v$ has exactly 2, 3, or 4 zeros



Stability Conditions

- ρ is stable $\Rightarrow \rho$ is supported on $\{0, \pi\}$ (for generic K/K_s)

$\Rightarrow$ Feasible solutions to original problem almost always recoverable



Stability Conditions

⇒ Binarized fixed points stable when:

$$K/K_s \leq 2$$

$$m \in \{0,1\},$$

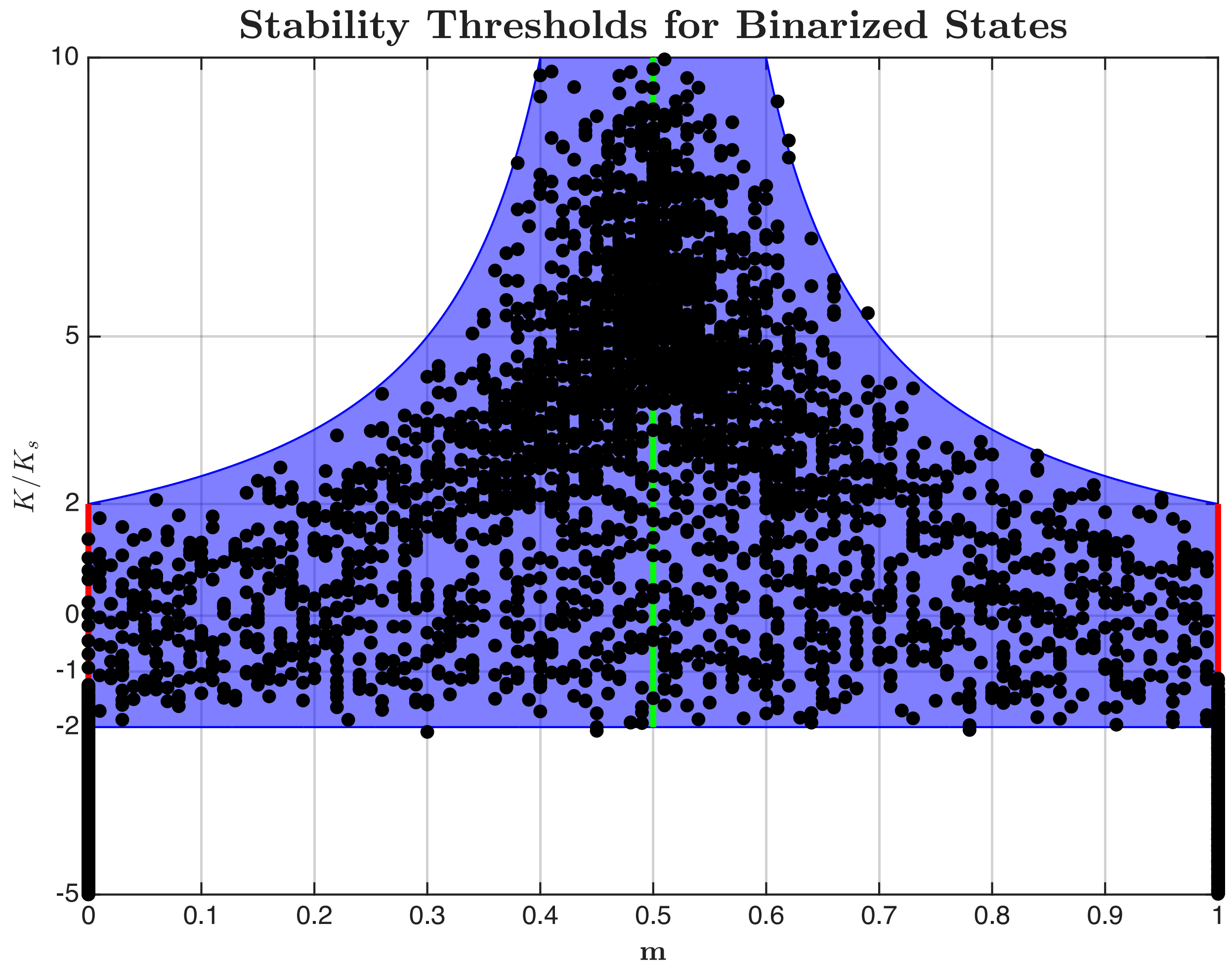
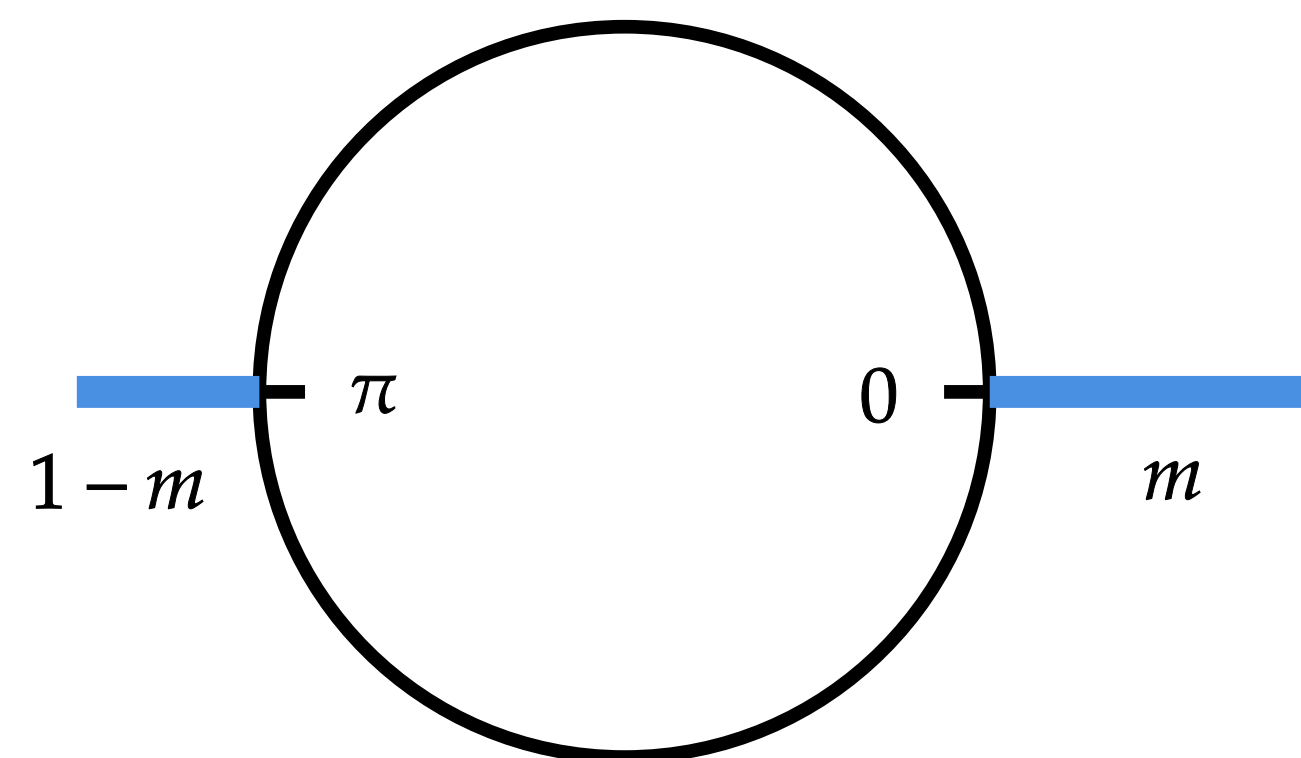
$$-2 \leq K/K_s$$

$$m = 0.5,$$

$$-2 \leq K/K_s \leq \frac{2}{|2m-1|} \quad \text{otherwise,}$$

where $m \in [0,1]$ is mass fraction at 0

⇒ Threshold for binarization is $-2 \leq K/K_s$



Looking Forward

- Big Question: How far can the analysis be extended?
- More general OIM models:

$$\dot{\theta}_i = \sum_j \underbrace{A_{ij}}_{\text{structured}} \sin(\theta_i - \theta_j) - \underbrace{K_i(t)}_{\text{free}} \sin(2\theta_i) + \underbrace{w_i(t)}_{\text{noise}}, \quad \theta_i(0) \sim \underbrace{X_i}_{\text{free}}$$

- Do these models retain gradient flow structures? Are they tractable?
- What is the relationship between frequency and quality of fixed points?
- How to choose free parameters to get best performance?

Thanks for watching! Questions?



Link to Paper



Personal Website