Optimal Combined Motion and Assignments with Continuum Models

Max Emerick[1] Stacy Patterson[2] Bassam Bamieh[1]

[1]University of California Santa Barbara
[2]Renssellaer Polytechnic Institute

July 6, 2022

NecSys 22

Introduction and Motivation

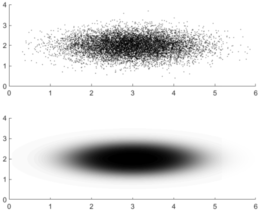
Autonomous swarms being developed for many applications:

Large swarms robust and efficient but hard to manage:

Discrete Models

$\rightarrow$

Continuum Models



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Approach and Contributions

The Problem:

- Motion planning/control for large swarms
- Based on **continuum model**
- Uses tools from **optimal transport theory** (assignments)

Related Work:

- Bandyopadhyay et. al. [1]
- Krishnan and Martínez [2]
- Inoue et. al. [3]

Limited objectives/constraints $\rightarrow$ **optimal control theory** (motion)

Key Contributions:

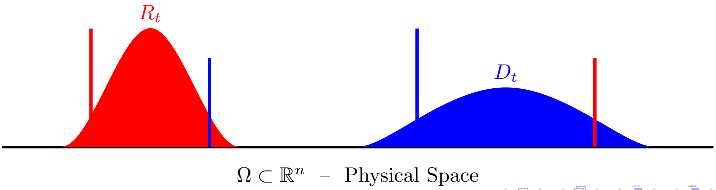
- Introduce novel model for swarm control
- Reparameterization and analytic solution in 1D

Demand and Resource Distributions

- Demand = known signal, requires services
- Resource = state of swarm, provides services
- Example: drones surveying wildfire

$$D_t(x) = d(x,t) + \sum_{k=1}^{N_d} d_k(t)\,\delta\big(x - \gamma_k(t)\big)$$

$$R_t(x) = \underbrace{r(x,t)}_{\text{continuous component}} + \underbrace{\sum_{k=1}^{N_r} r_k(t)\,\delta\big(x - \eta_k(t)\big)}_{\text{discrete component}}$$

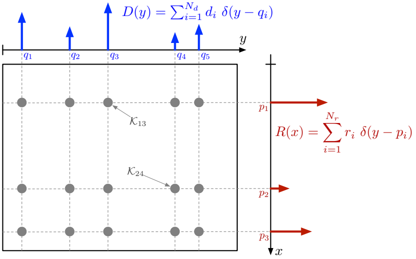


Assignment Kernel

Kantorovich Problem (Optimal Transport)

$$\underbrace{\mathcal{W}_2^2(R_t, D_t)}_{\text{assignment cost}} = \min_{\mathcal{K}} \int_{\Omega\times\Omega} \underbrace{|y-x|^2}_{\text{pairwise cost}} \underbrace{\mathcal{K}(x,y)}_{\text{assignment kernel}} \; dx\, dy \qquad \text{s.t.} \qquad \underbrace{\begin{array}{c} \Pi_y \, \mathcal{K}_t = R_t \\ \Pi_x \, \mathcal{K}_t = D_t \end{array}}_{\text{marginalization}}$$

- Assignment is **joint distribution** between resource and demand
- Infinite-dimensional linear program
- $\mathcal{W}_2$ = **2-Wasserstein distance**



Discrete Assignment Kernel

Dynamic Model

- Want resources close to demand
- Control = velocity field V
- Dynamics given by **transport equation**:

$$\partial_t R(x,t) \;=\; -\nabla \cdot \big(V(x,t)\; R(x,t) \big)$$

- Note: same location $\Rightarrow$ same velocity
- Motion cost $= \int_\Omega \|V_t(x)\|^2 R_t(x)\, dx$ $\quad$ ($\sim$ "drag losses")

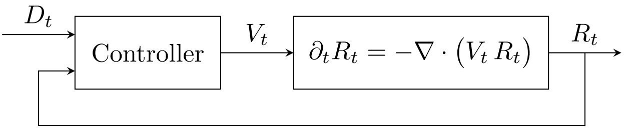
Optimal Control Problem

Total cost expressed by objective function:

$$\mathcal{J} \;=\; \int_0^T \left(\underbrace{\mathcal{W}_2^2\left(R_t, D_t\right)}_{\substack{\text{assignment}\\ \text{cost}}} \;+\; \alpha \underbrace{\int_\Omega \left\| V_t(x) \right\|^2 R_t(x)\, dx}_{\text{motion cost}} \right) dt$$

Competing

Problem: Find controller which minimizes $\mathcal{J}$



- Nonlinear infinite-dimensional optimal control problem in general
- In 1D, can transform to infinite-dimensional LQ tracking problem

Mathematical Background

Cumulative Distribution and Quantile Functions

Distribution	$\mu(x)$	
CDF	$F_\mu(x) := \int_{-\infty}^{x} \mu(\xi)\, d\xi$	function inverses
Quantile	$Q_\mu(z) := \inf\{x : F_\mu(x) \geq z\}$	

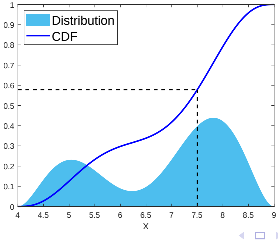
Wasserstein Distance in 1D (well-known result, e.g. [4])

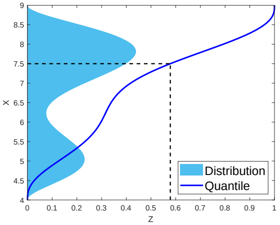
$$\mathcal{W}_2^2(\mu,\nu) = \int_0^1 (Q_\nu(z) - Q_\mu(z))^2\, dz$$

Distributions with $\mathcal{W}_2$

↕ isometric

Quantile functions with L^2





Equivalent Problem

Original Problem

- Objective:

$$\mathcal{J} \; = \; \int_0^T \left(\mathcal{W}_2^2 \left(R_t, D_t \right) \; + \; \alpha \int_\Omega \| V_t(x) \|^2 R_t(x) \, dx \right) dt$$

- Dynamics: $\partial_t R_t(x) = -\nabla \cdot \left(V_t(x) \; R_t(x) \right)$
- Implicit constraint: same location $\Rightarrow$ same velocity

Equivalent Problem

- Objective:

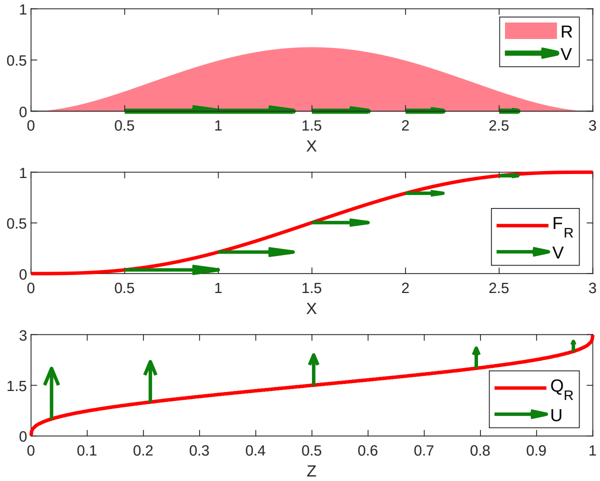
$$\mathcal{J} \; = \; \int_0^T \int_0^1 \left(\left(Q_D(z,t) - Q_R(z,t) \right)^2 \; + \; \alpha \, U^2(z,t) \right) dz \, dt$$

- Dynamics: $\partial_t Q_R(z,t) \; = \; U(z,t)$
- Explicit constraint: $\partial_z Q_R(z,t) = 0 \; \Rightarrow \; \partial_z U(z,t) = 0$

Equivalent Problem

Original Problem → Dynamics

Equivalent Problem → Dynamics



1D Problem: Solution

- Simplifying feature: dynamics and objective **decoupled in space**

$$\mathcal{J} = \int_0^1 \left(\int_0^T \big(Q_D(z,t) - Q_R(z,t)\big)^2 + \alpha U^2(z,t)\, dt \right) dz$$

$$\text{s.t.} \quad \partial_t Q_R(z,t) \; = \; U(z,t) \qquad\qquad \text{(decoupled in z)}$$

- Solution at different z coupled only through input constraint

$$\partial_z Q_R(z,t) = 0 \quad \Rightarrow \quad \partial_z U(z,t) = 0$$

- Equivalent: solve a scalar LQ problem for each decoupled region

The Scalar LQ Tracking Problem

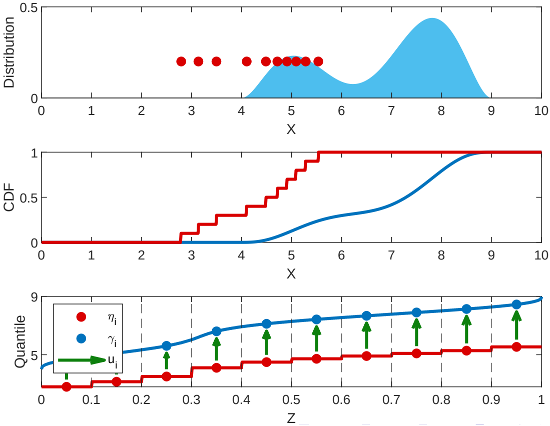
Scalar Linear-Quadratic Tracking Problem

$$\bar{u} = \operatorname*{argmin}_{u} \int_0^T \big(\gamma(t) - \eta(t)\big)^2 + \alpha u^2(t)\, dt \qquad \text{s.t.} \quad \dot{\eta} = u$$

η = transformed resource

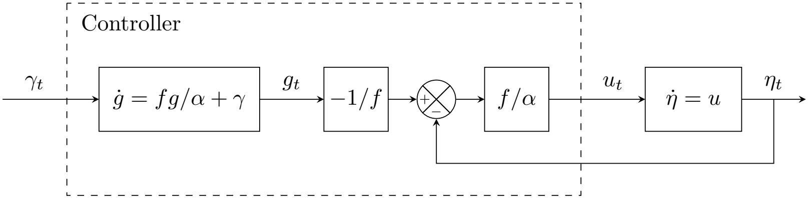
u = control input

γ = transformed demand



Solution to Scalar LQ Problem

Optimal solution has feedback and feedforward term:



Controller parameters:

$$\dot{f}(t) = f^2(t)/\alpha - 1 \qquad f(T) = 0$$
$$\dot{g}(t) = f(t)\,g(t)/\alpha + \gamma(t) \qquad g(T) = 0$$

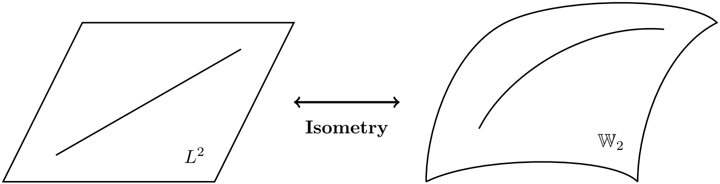
- General solution exists
- Two interesting cases: static and periodic

Solution: Static Demand

Static demand $\Rightarrow$ closed-form solution for trajectories:

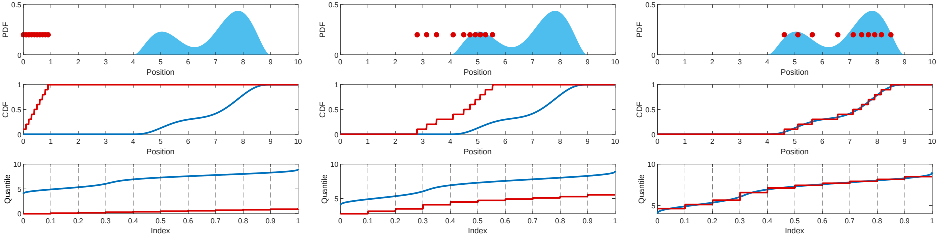
$$\boxed{\eta(t) \;=\; \Phi(t)\,\eta_0 + (1-\Phi(t))\,\gamma}$$

- Optimal trajectory is linear interpolation between initial state η_0 and transformed demand γ
- Straight lines in L^2 map to **geodesics** in Wasserstein space



Simulation Results: Static Demand

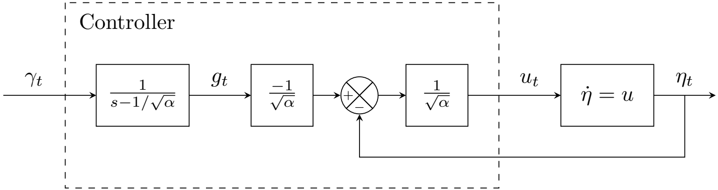
- Discrete resource distribution, 10 identical agents
- Continuous static demand distribution
- Time horizon $T = 10$, weighting parameter $\alpha = 2$
- Approaches nearest reachable distribution



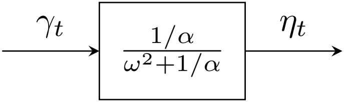
Initial Conditions — Intermediate Conditions — Final Conditions

Solution: Periodic Demand

Steady-state response as $T \to \infty$:

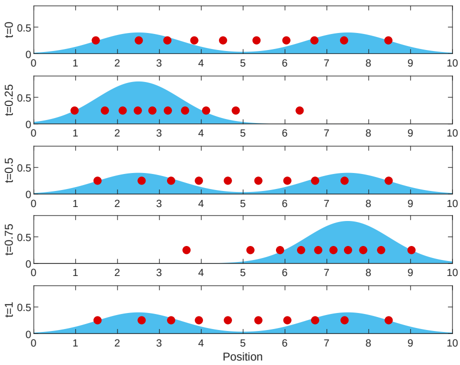


Overall transfer function:

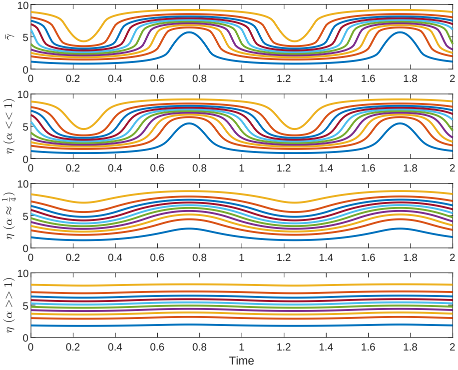


- Second-order low-pass filter with $f_c = 1/\sqrt{\alpha}$
- Transformed resource and demand perfectly in-phase

- Discrete resource distribution, 10 identical agents
- Continuous periodic demand distribution @ $f = 1$ hz
- Simulated at three different values of α



Resource and Demand Distributions



Transformed Resource and Demand

Conclusion

Summary:

- Presented novel model for control of large swarms
- Demonstrated reparameterization and analytic solution in 1D case
- Shared simulation results for static and periodic cases

Future Work:

- Investigate higher dimensions
- Develop causal optimal controller
- Apply towards distributed control methods

Thanks for Watching!

Questions?

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