Control of Densities in Wasserstein Space

Max Emerick

University of California Santa Barbara

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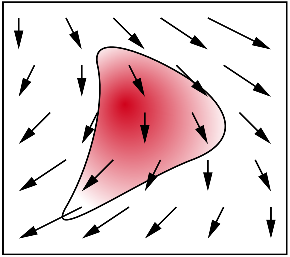
Control of Densities Arises in Many Contexts

- Spatially distributed systems
- Ensemble systems
- Stochastic systems
- Generative artificial intelligence



Individual Evolution Equation

$$\dot{x} = f(x, u)$$

$$\Updownarrow$$

Collective Evolution (Transport) Equation

$$\partial_t \psi(x, t) = -\nabla \cdot \big(f(x, u)\, \psi(x, t)\big)$$

Spatially Distributed Systems

- Physical "clouds" of particles/agents approximated by density
- Interactions $\rightarrow$ more complex dynamics
- Inputs may be constrained

Velocity Control

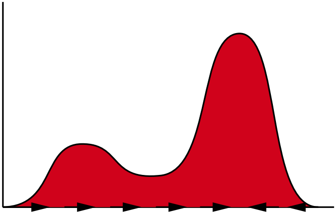
$$\dot{x} = v \qquad \Rightarrow \qquad \partial_t \psi(x,t) = -\nabla \cdot \big(v(x,t)\, \psi(x,t)\big)$$

Acceleration Control

$$\ddot{x} = a \qquad \Rightarrow \qquad \partial_t \psi(x,v,t) = -\nabla \cdot \left(\left[\begin{smallmatrix} v \\ a(x,t) \end{smallmatrix}\right] \psi(x,v,t)\right)$$

Ensemble Systems

- Large population of subsystems
- State distribution is statistical
- Different subsystems may or may not have identical dynamics



Identical Dynamics

$$\dot{x} = f(x, u) \qquad \Rightarrow \qquad \partial_t \, \psi(x, t) = -\nabla \cdot \big(f(x, u(t)) \, \psi(x, t) \big)$$

Non-Identical Dynamics

$$\dot{x} = f_\alpha(x, u) \qquad \Rightarrow \qquad \partial_t \, \psi(x, \alpha, t) = -\nabla \cdot \big(f_\alpha(x, u(t)) \, \psi(x, \alpha, t) \big)$$

Stochastic Systems

- Single system evolving stochastically
- State distribution is probabilistic

Ito SDE/Langevin Equation

$$x\,dt = f(x,u)\,dt + g(x)\,dW$$

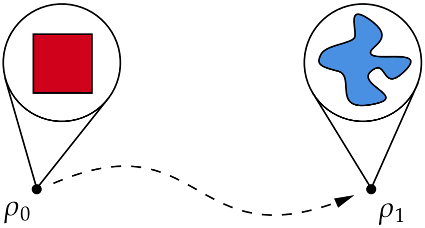
$$\Updownarrow$$

Fokker-Planck/Forward Kolmogorov Equation

$$\begin{aligned}\partial_t\,\psi(x,t) &= -\nabla\cdot\big(f(x,u)\,\psi(x,t)\big) + \nabla\cdot\big(D(x)\,\nabla\psi(x,t)\big)\\ &= -\nabla\cdot\left(\underbrace{\left(f(x,u) - \tfrac{D(x)\nabla\psi(x,t)}{\psi(x,t)}\right)}_{\text{density-dependent velocity}}\,\psi(x,t)\right)\end{aligned}$$

Generative Artificial Intelligence

- Generative models conceptualized as "sampling from distribution"
- Aim to learn transformation b/w reference density ρ_0 and target ρ_1
- Modeling with transport/diffusion processes popular



- Learning dynamics from data is related problem
- Key challenges on numerical/estimation/data-driven side

Density Control Problems

- Density control problems arise in many settings, rich class
- Often interested in feedback control

Feedback Control Problem

Find $K : (x, t, \psi, ...) \mapsto u$ satisfying $\langle$additional constraints$\rangle$ such that

$$\begin{cases} \partial_t \psi(x,t) = -\nabla \cdot \big(f(x,u)\,\psi(x,t)\big) \\ u = K(x,t,\psi,...) \end{cases}$$

has $\langle$desired properties$\rangle$

- Complicated velocity fields provide interesting challenges:
  - Density-dependence (self-interaction, stochasticity)
  - Constraints (underactuation, incompressibility, locality, etc.)
- Numerical, estimation, data-driven challenges too

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Motivation:

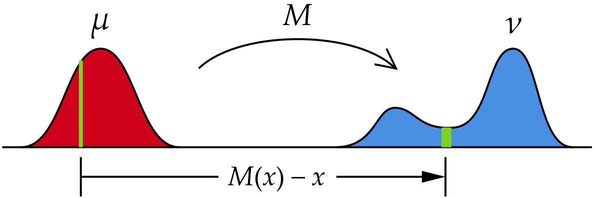
Wasserstein Distance Lets Us Compare Densities

Preliminaries: Monge Problem

Monge Problem[1]

$$\inf_M \int_\Omega \left\|M(x) - x\right\|_2^2 \mu(x)\, dx \qquad \text{s.t.} \qquad M_{\#}\mu = \nu$$

- $\#$ denotes **measure pushforward**
- Densities μ, ν must be normalized
- Minimizer $\bar{M}_{\mu\to\nu}$ is **optimal transport map**



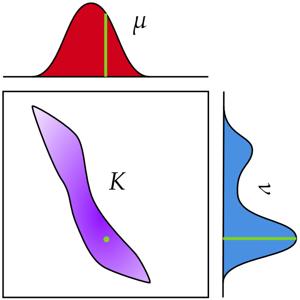
[1] Gaspard Monge. "Mémoire sur la théorie des déblais et des remblais". In: *Mem. Math. Phys. Acad. Royale Sci.* (1781), pp. 666–704.

Preliminaries: Kantorovich Problem

Kantorovich Problem[2]

$$\inf_K \int_{\Omega\times\Omega} \|x-y\|_2^2 \, K(x,y)\, dx\, dy \qquad \text{s.t.} \qquad \begin{array}{l} \int K(\cdot,y)\, dy = \mu \\ \int K(x,\cdot)\, dx = \nu \end{array}$$

- Minimizer $\bar{K}$ is **optimal transport plan**
- Minimum is squared **2-Wasserstein distance** $\mathcal{W}_2^2(\mu,\nu)$
- Infinite-dimensional linear program, has dual formulation
- Works for discrete densities



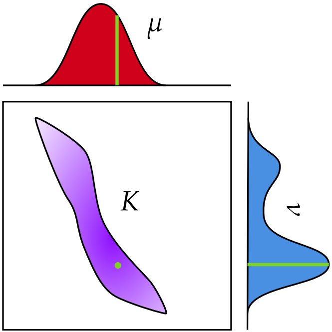
[2] Leonid V Kantorovich. "On the translocation of masses". In: *Dokl. Akad. Nauk. USSR (NS)*. vol. 37. 1942, pp. 199–201.

Kantorovich Problem (Statistical Interpretation)

Kantorovich Problem (Statistical Interpretation)

$$\inf_K \mathbb{E}_K \left[d^2(X, Y)\right] \qquad \text{s.t.} \qquad K_X = \mu,\ K_Y = \nu$$

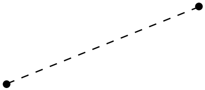
- $X \sim \mu$, $Y \sim \nu$ are random variables
- $K = K_{X,Y}$ is joint distribution
- d^2 is squared Euclidean distance



Wasserstein Distance as Extension of Ground Metric

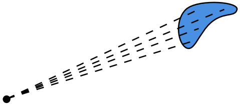
- Both densities Dirac masses:

$$\mathcal{W}_2(\delta_x, \delta_y) = d(x, y)$$



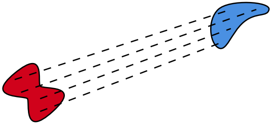
- One density Dirac, one arbitrary:

$$\mathcal{W}_2(\delta_x, \nu) = \sqrt{\mathbb{E}_\nu \left[d^2(x, y) \right]}$$



- Both densities arbitrary $\rightarrow$ depends on assignment:

$$\mathcal{W}_2(\mu, \nu) = \inf_K \sqrt{\mathbb{E}_K \left[d^2(x, y) \right]}$$



Preliminaries: Dynamic Optimal Transport

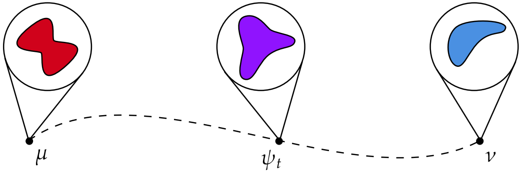
Dynamic Optimal Transport[3]

$$\partial_t \psi(x,t) = -\nabla \cdot \big(v(x,t)\,\psi(x,t)\big)$$

$$\inf_{\psi, v} \quad J = \int_0^1 \left(\int_\Omega \|v(x,t)\|_2^2 \, \psi(x,t)\, dx \right) dt$$

$$\psi(\cdot,0) = \mu(\cdot), \quad \psi(\cdot,1) = \nu(\cdot)$$

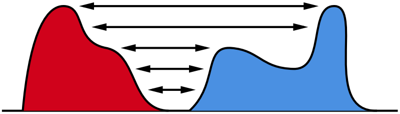
- Optimal state-transfer control problem
- Gives optimal path from μ to ν (not just assignment)



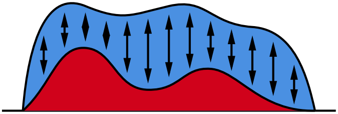
[3] Jean-David Benamou and Yann Brenier. "A computational fluid mechanics solution to the Monge-Kantorovich mass transfer problem". In: *Numerische Mathematik* 84 (2000), 375–393.

Wasserstein Distance in Density Control

- One of many ways to compare densities
- Often natural (metric on ground space, mass preserved)
- "Horizontal sense" as opposed to "vertical sense"
- Connections with optimal control, transport dynamics
- Useful theoretical properties:
  - Riemannian structure
  - PDEs as gradient flows



W_2



L^2, KL, TV

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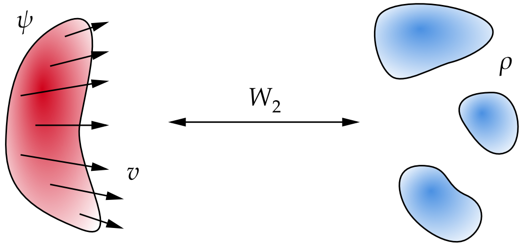
Example: Tracking Control for Swarms

- Problem: design controller so that swarm ψ tracks reference ρ
- Want stability, asymptotic tracking of constant signals

Dynamics: Velocity Control

$$\dot{x} = v \qquad \Rightarrow \qquad \partial_t \psi_t = -\nabla \cdot \left(v_t \psi_t\right)$$

$$\left(\text{Notation:} \qquad \psi_t := \psi(\cdot, t)\right)$$



Approach

- Treat as regulation problem with changing setpoint ρ
- Consider constant-ρ case first
- Optimal control-based design

Problem Statement

$$\partial_t \psi_t = -\nabla \cdot (v_t \psi_t)$$

$$\inf_{\psi, v} \quad J = \int_0^\infty \Big(\underbrace{\mathcal{W}_2^2(\psi_t, \rho)}_{\text{tracking error}} + \underbrace{\alpha \, \|v_t\|^2_{L^2(\psi_t)}}_{\text{control effort}} \Big) dt$$

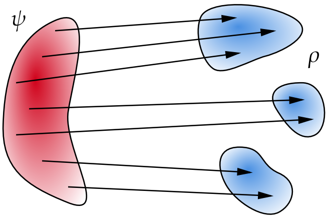
$$\left(\text{Notation:} \qquad \|v_t\|^2_{L^2(\psi_t)} := \int_\Omega \|v(x,t)\|_2^2 \, \psi(x,t) \, dx \right)$$

Preview of Solution

Optimal Controller

$$v_t = \frac{1}{\sqrt{\alpha}}(\bar{M}_{\psi\to\rho} - \mathcal{I})$$

- Optimal transport map $\bar{M}_{\psi\to\rho}$ gives particles "assignments"
- Particles move towards assigned particles at rate $dist/\sqrt{\alpha}$



Constant ρ:

https://www.dropbox.com/scl/fo/v605jq5n5dcprdfrxt4rh/h?dl=0&e=1&preview=static.mp4&rlkey=nj6zc9ssk38abc99javzvb0tr

Approach

We solve this problem by leveraging two big results:

1. The **Eulerian-Lagrangian correspondence**
2. The **Riemannian structure** of the Wasserstein space

Main Difficulty

Problem

$$\partial_t \psi_t = -\nabla \cdot (v_t \psi_t)$$

$$\inf_{\psi, v} \quad J = \int_0^\infty \left(\mathcal{W}_2^2(\psi_t, \rho) + \alpha \, \|v_t\|^2_{L^2(\psi_t)} \right) dt$$

$\rightarrow$ Necessary Conditions
$$\begin{cases} \partial_t \psi_t = -\nabla \cdot (\nabla \lambda_t \psi_t) \\ \partial_t \lambda_t = -\frac{1}{2} \|\nabla \lambda_t\|^2 + \frac{1}{2\alpha^2} \frac{\delta}{\delta \psi_t} \mathcal{W}_2^2(\psi_t, \rho) \end{cases}$$

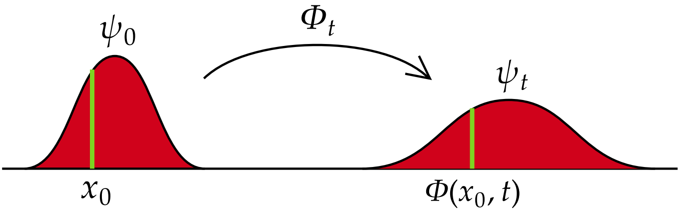
- Nonlinear two-point boundary-value PDE
- $\mathcal{W}_2^2$ requires solving optimization problem

Key Idea:

Problem is Simpler When Formulated Differently

Coordinate Change

- Dynamics of particles much simpler than those of densities
- Want to rewrite system in terms of particle dynamics
- Encode positions of particles in map Φ
- Reparameterize problem in terms of maps



Dynamics of Φ

$$\dot{x} = v(x) \qquad \rightarrow \qquad \partial_t \Phi(x,t) = v\big(\Phi(x,t), t\big)$$

Eulerian-Lagrangian Correspondence

Eulerian Representation

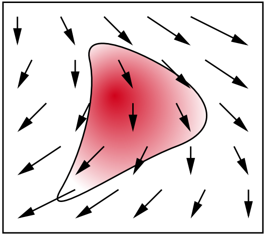
- Evolve densities over fixed locations
- Dynamics given by **transport equation:** $\partial_t \psi_t = -\nabla \cdot (v_t \psi_t)$

Lagrangian Representation

- Evolve locations of fixed particles
- Dynamics given by **flow equation:** $\partial_t \Phi(x,t) = v\big(\Phi(x,t), t\big)$

Correspondence given by pushforward:

$$\psi_t = \big[\Phi_t\big]_{\#} \psi_0$$



Transformed Quantities

Dynamics

$$\partial_t \psi_t = -\nabla \cdot (v_t \psi_t) \qquad \to \qquad \partial_t \Phi_t = v_t(\Phi_t) =: u_t$$

Control Effort

$$\|v_t\|^2_{L^2(\psi_t)} \qquad \to \qquad \|u_t\|^2_{L^2(\psi_0)}$$

Tracking Error

$$\mathcal{W}_2^2(\psi_t, \rho) \qquad \to \qquad \left\|\bar{M}_{\psi_t \to \rho} - \mathcal{I}\right\|^2_{L^2(\psi_t)} = \left\|\bar{M}_{\psi_0 \to \rho} - \Phi\right\|^2_{L^2(\psi_0)}$$

- Can be formalized using the framework of Riemannian geometry[4]

[4] Felix Otto. "The geometry of dissipative evolution equations: the porous medium equation". In: (2001).

Equivalent Problem

Original Problem

$$\partial_t \psi_t = -\nabla \cdot (v_t \psi_t)$$

$$\inf_{\psi, v} \quad J = \int_0^\infty \left(\mathcal{W}_2^2(\psi_t, \rho) + \alpha \, \|v_t\|^2_{L^2(\psi_t)} \right) dt$$

$$\updownarrow$$

Equivalent Problem

$$\partial_t \Phi_t = u_t$$

$$\inf_{\Phi, u} \quad J = \int_0^\infty \left(\left\| \bar{M}_{\psi_0 \to \rho} - \Phi_t \right\|^2_{L^2(\psi_0)} + \alpha \, \|u_t\|^2_{L^2(\psi_0)} \right) dt$$

Infinite-dimensional linear-quadratic problem!

Solution to Equivalent Problem

Equivalent Problem

$$\partial_t \Phi_t = u_t$$

$$\inf_{\Phi,u} \quad J = \int_0^\infty \left(\left\| \bar{M}_{\psi_0 \to \rho} - \Phi_t \right\|^2_{L^2(\psi_0)} + \alpha \left\| u_t \right\|^2_{L^2(\psi_0)} \right) dt$$

$\rightarrow$ Necessary Conditions $\begin{cases} \partial_t \Phi_t = -\frac{1}{\alpha} \Lambda_t \\ \partial_t \Lambda_t = \bar{M}_{\psi_0 \to \rho} - \Phi_t \end{cases}$

$\rightarrow$ (Details here[5]) $\rightarrow$

$$u_t = \frac{1}{\sqrt{\alpha}} (\bar{M}_{\psi_0 \to \rho} - \Phi_t) \quad \rightarrow \quad v_t = \frac{1}{\sqrt{\alpha}} (\bar{M}_{\psi_t \to \rho} - \mathcal{I})$$

[5] Max Emerick and Bassam Bamieh. "Continuum Swarm Tracking Control: A Geometric Perspective in Wasserstein Space". In: *2023 62nd IEEE Conference on Decision and Control (CDC)*. 2023, pp. 1367–1374.

Properties of Controller

Optimal Controller

$$v_t = \frac{1}{\sqrt{\alpha}}(\bar{M}_{\psi_t \to \rho} - \mathcal{I})$$

Optimal Trajectory

$$\psi_t = \left[(1 - \sigma(t))\,\mathcal{I} + \sigma(t)\,\bar{M}_{\psi_0 \to \rho}\right]_{\#}\,\psi_0$$

- Can show that ψ_t follows **geodesic** from ψ_0 to ρ
- Optimal control determines time schedule $\sigma(t)$

Constant ψ:

https://www.dropbox.com/scl/fo/v605jq5n5dcprdfrxt4rh/h?dl=0&e=1&preview=static.mp4&rlkey=nj6zc9ssk38abc99javzvb0tr

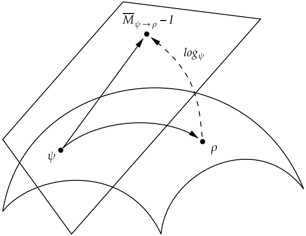
Time-Varying Case

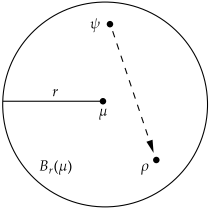
- Can apply same controller when ρ is changing

Controller

$$v_t = \frac{1}{\sqrt{\alpha}}(\bar{M}_{\psi_t \to \rho_t} - \mathcal{I})$$

- Can be interpreted as proportional controller on manifold[6]
- Can be shown to be stable in $\mathcal{W}_2$-BIBO-sense





[6] Simone Fiori. "Extension of PID regulators to dynamical systems on smooth manifolds (M-PID)". In: *SIAM Journal on Control and Optimization* 59.1 (2021), pp. 78–102.

Simulations

Time-Varying ρ:

https://www.dropbox.com/scl/fo/v605jq5n5dcprdfrxt4rh/h?dl=0&e=2&preview=constant_velocity.mp4&rlkey=nj6zc9ssk38abc99javzvb0tr

https://www.dropbox.com/scl/fo/v605jq5n5dcprdfrxt4rh/h?dl=0&e=2&preview=fading.mp4&rlkey=nj6zc9ssk38abc99javzvb0tr

Takeaways

- Density control problems are a rich class of problems
- Wasserstein distance is natural and useful in many instances
- Eulerian-Lagrangian correspondence and Riemannian structure are key tools for solving density control problems in Wasserstein space

Thanks to My Collaborators

Bassam Bamieh

Jared Jonas

Thanks for Watching!

Questions?

Appendix: Eulerian-Lagrangian Correspondence

- Want to pose problem in terms of flow maps $\Phi \in \mathbb{M}$
- Motivated by correspondence $\rho_t = [\Phi_t]_{\#}\, \rho_0$, define

$$\begin{aligned} \Pi : \mathbb{M} &\to \mathbb{W} \\ M &\mapsto M_{\#}\rho_0 \end{aligned}$$

- Equivalent dynamics $\rightarrow$ following diagram commutes:

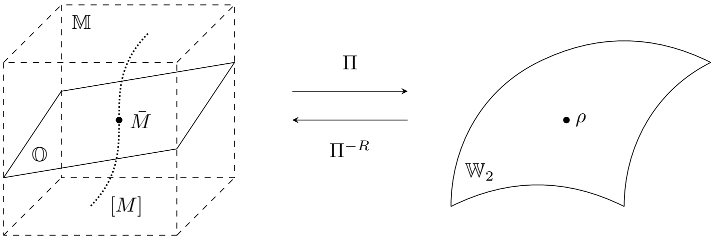
$$\begin{tikzcd} Id \arrow[r, "\partial_t \Phi = v \circ \Phi"] \arrow[d, "\Pi"'] & \Phi_t \arrow[d, "\Pi"] \\ \rho_0 \arrow[r, "\partial_t \rho = -\nabla \cdot (v\rho)"'] & \rho_t \end{tikzcd}$$

Appendix: Eulerian-Lagrangian Correspondence

- Problem: Π is not invertible
- Solution: define Π^{-R} using optimal transport map

$$\begin{aligned}\Pi^{-R} : \mathbb{W} &\to \mathbb{M} \\ \rho &\mapsto \bar{M}_{\rho_0 \to \rho}\end{aligned}$$

Densities in $\mathbb{W}_2$ 1-1 with OT maps in $\mathbb{O} \subset \mathbb{M}$:



Appendix: Eulerian-Lagrangian Correspondence

- Key fact: $\Pi : \mathbb{M} \to \mathbb{W}_2$ is a **Riemannian submersion**
- ($\mathbb{M}$ and $\mathbb{W}_2$ are Riemannian manifolds, and $D\Pi$ is an orthogonal projection onto each tangent space)
- This allows us to make the following identifications:

	$\mathbb{O}$		$\mathbb{W}_2$
Points:	M		ρ
Distance:	$\lVert M_{\rho_0 \to \rho} - \mathcal{I} \rVert_{L^2(\rho_0)}$	$\longleftrightarrow$	$\mathcal{W}_2(\rho, \rho_0)$
Dynamics:	$\partial_t \Phi = u$		$\partial_t \rho = -\nabla \cdot (v\rho)$
Speed:	$\lVert u \rVert_{L^2(\rho_0)}$		$\lVert v \rVert_{L^2(\rho)}$

Appendix: Stability of Controller

- Type of stability → $\mathcal{W}_2$-BIBO:

$$\psi_0, \psi_t \in B_r(\mu) \quad \forall t \qquad \Rightarrow \qquad \psi_t \in B_r(\mu) \quad \forall t$$

- Proof by contradiction: