Incompressible Optimal Transport and Applications in Fluid Mixing

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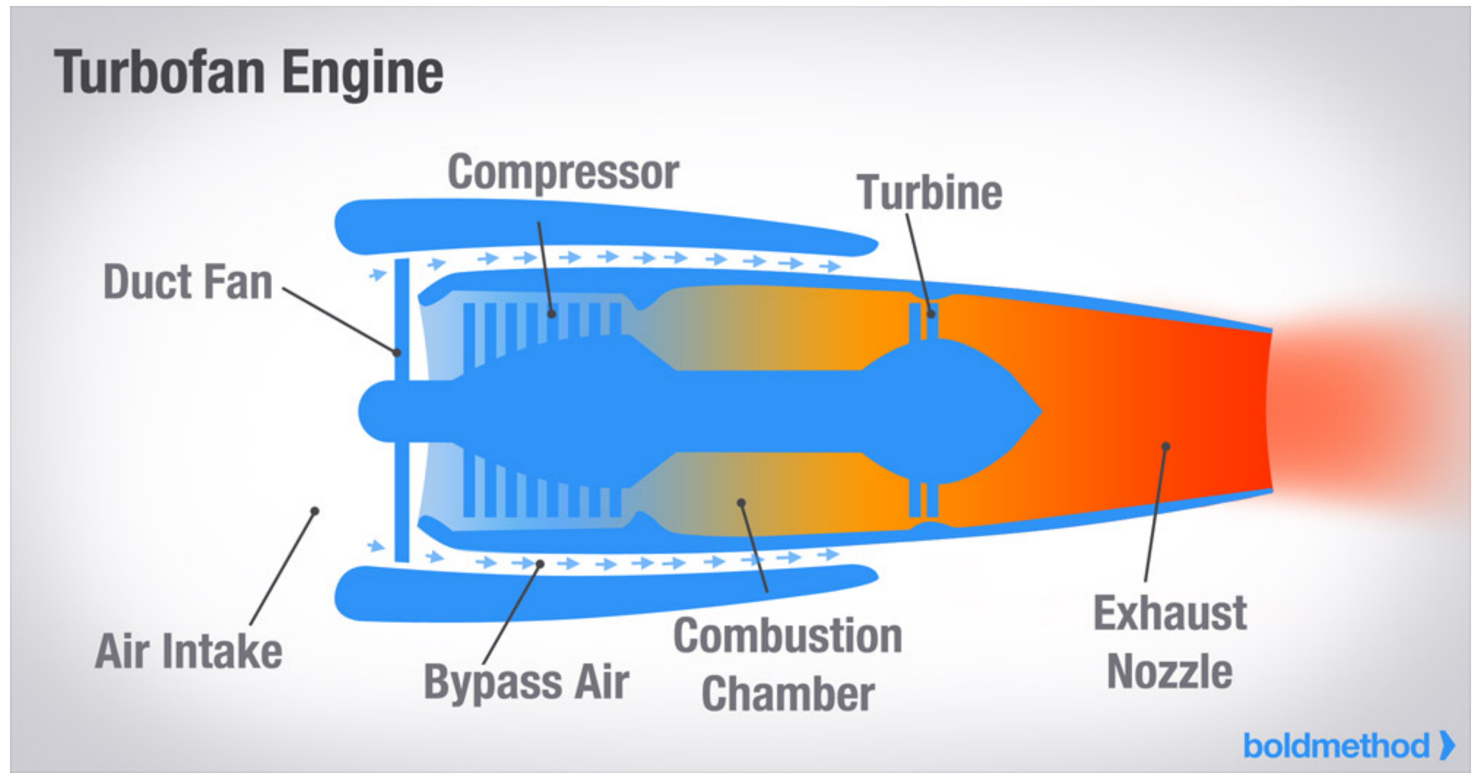
Fluid Mixing is All Around Us

Milk and Coffee

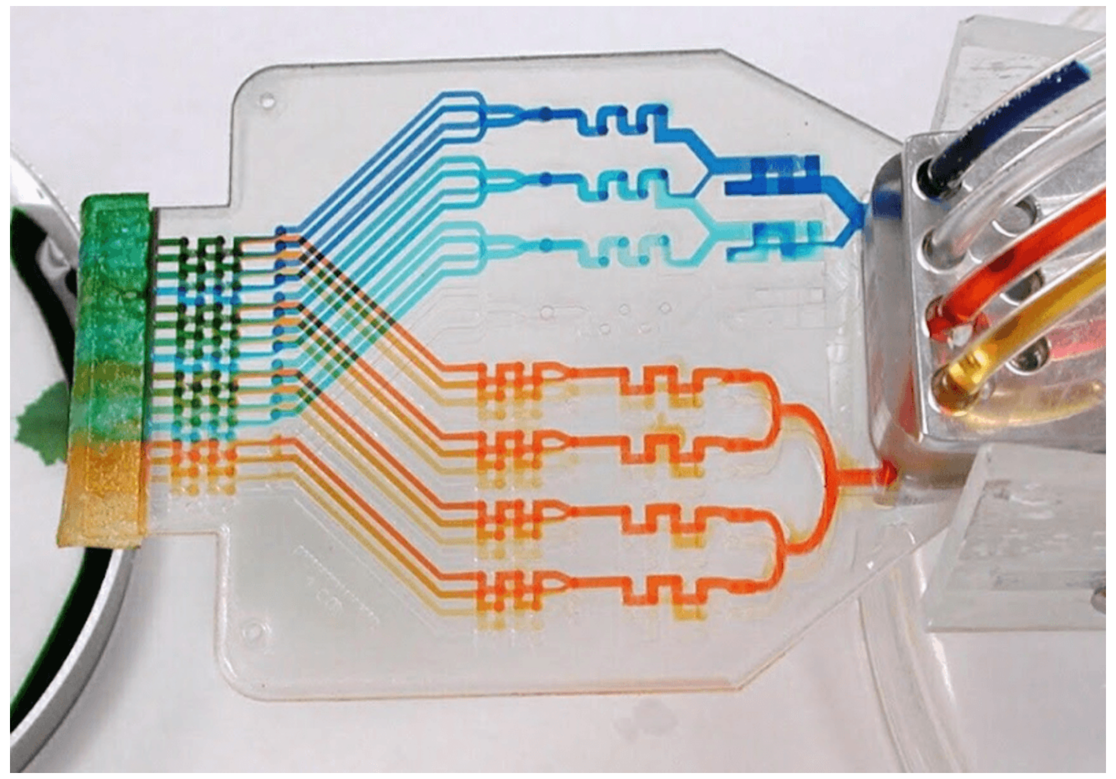
Chemical Processes

Biological Systems

Food Processing



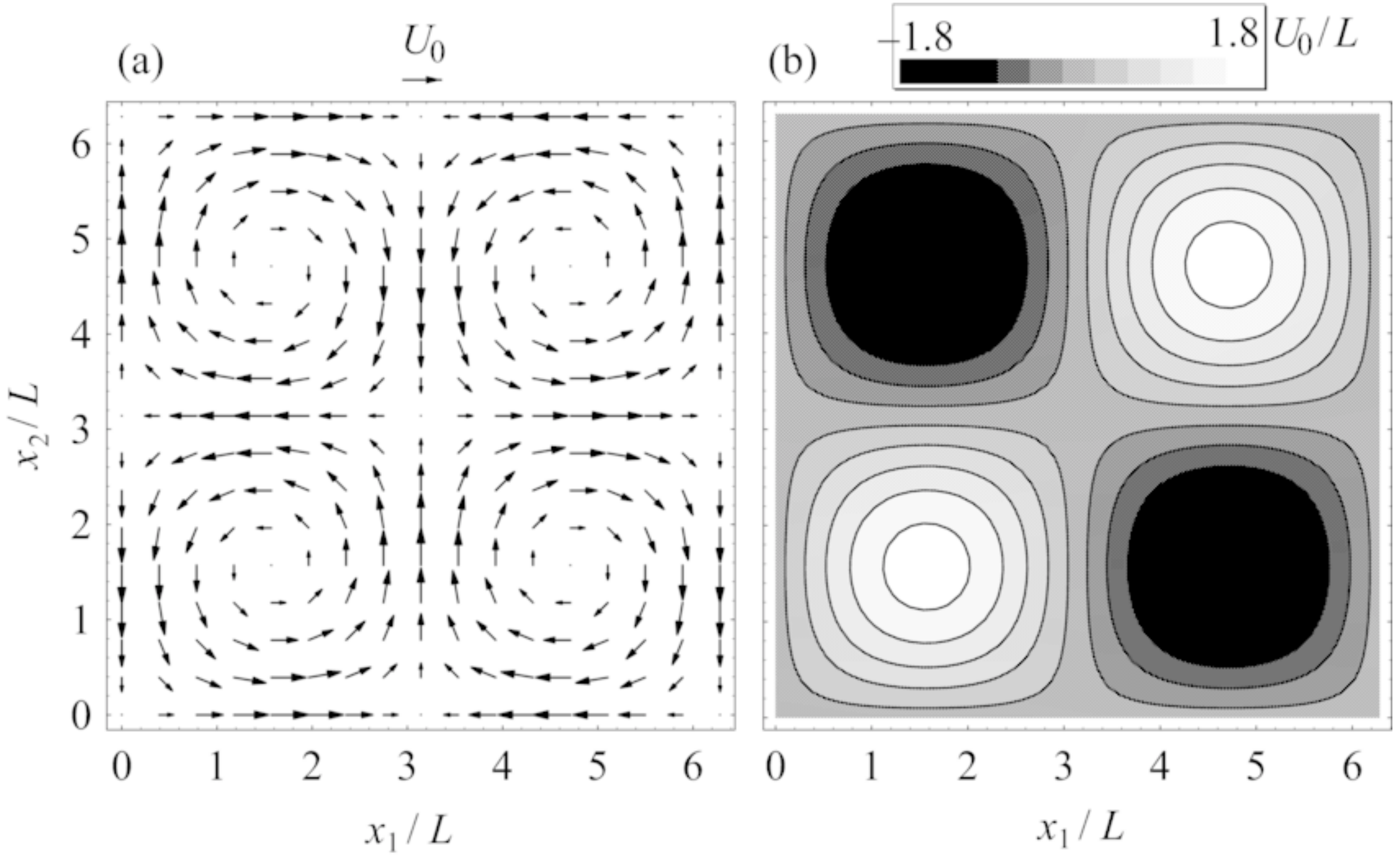
Combustion Engines

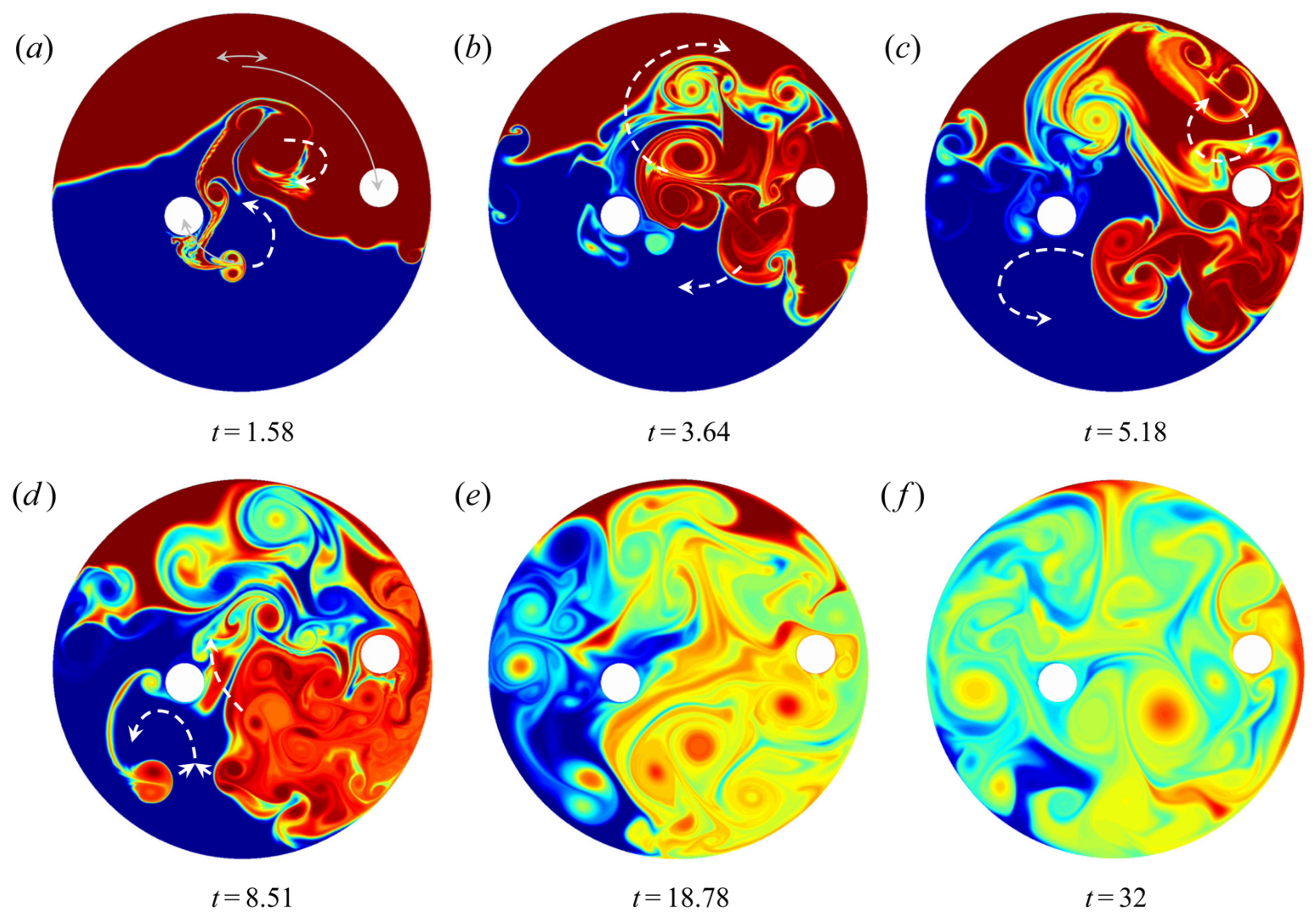


Microfluidic Devices

History of Fluid Mixing Problem

- **1870s-1940's: turbulence** (Boussinesq, Reynolds, Taylor, Kolmogorov, …)
- **1960s-1990s: chaos** (Arnold, Aref, Ottino, …)
- **2000s - Present: optimal mixing**
  - **Mixing rates** (Constantin, Thiffeault, Doering, Bressan, Kiselev, Seis, …)
  - **Control for mixing** (Mezic, Aamo, Krstic, Cortelezzi, Hu, Schmid, …)





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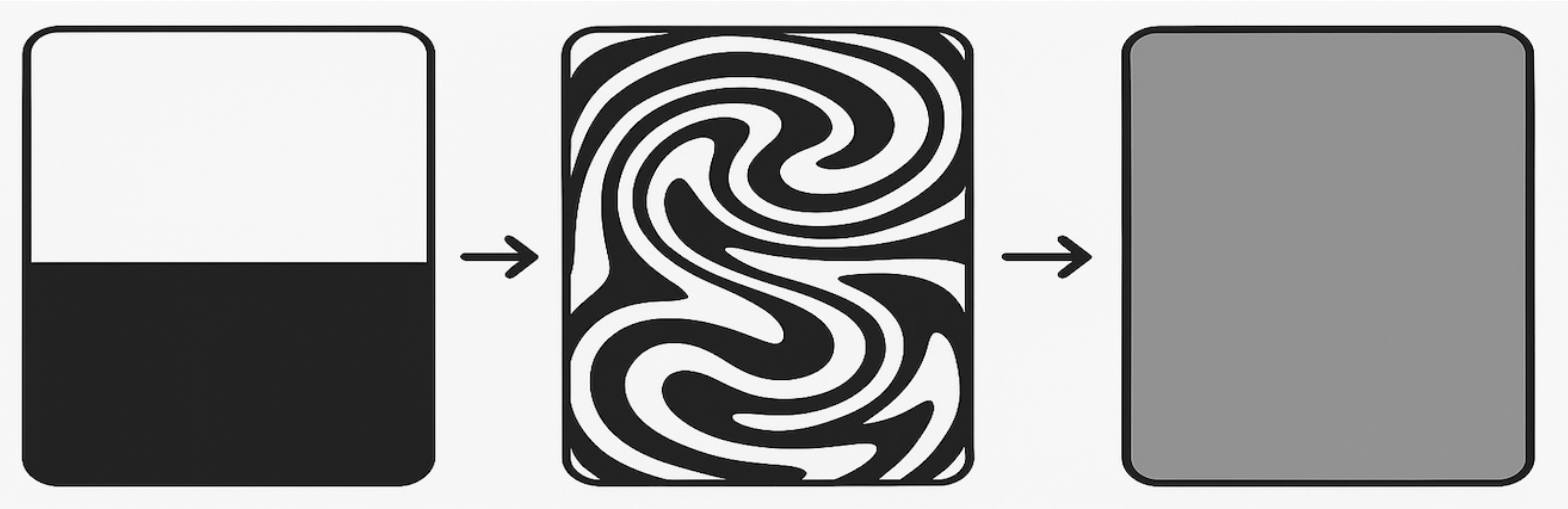
Open question: precise characterization of optimal mixing fields

Problem Formulation



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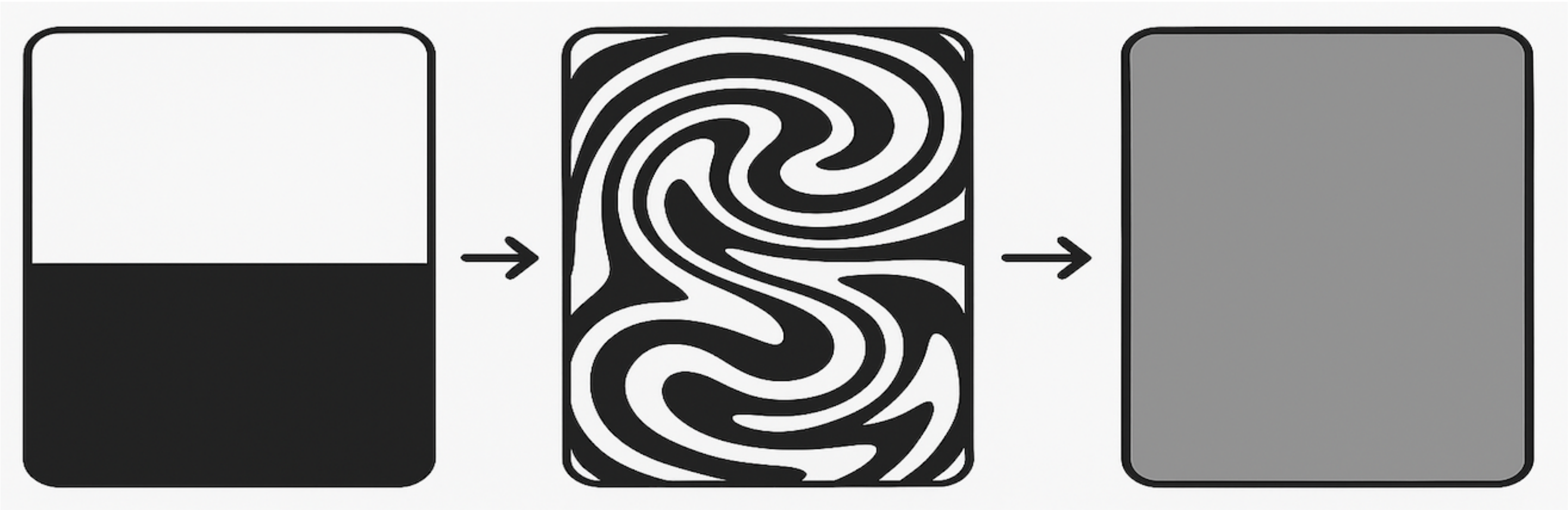

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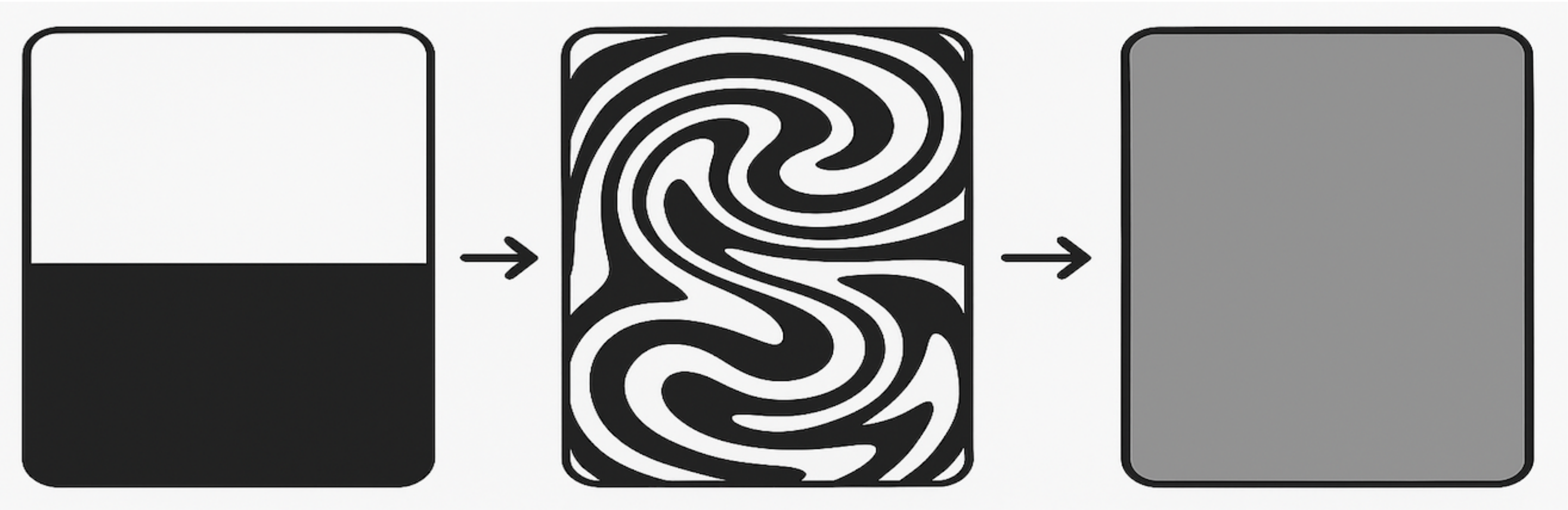
- Advective regime $\rightarrow \partial_t \rho = -\nabla \cdot (\rho v)$



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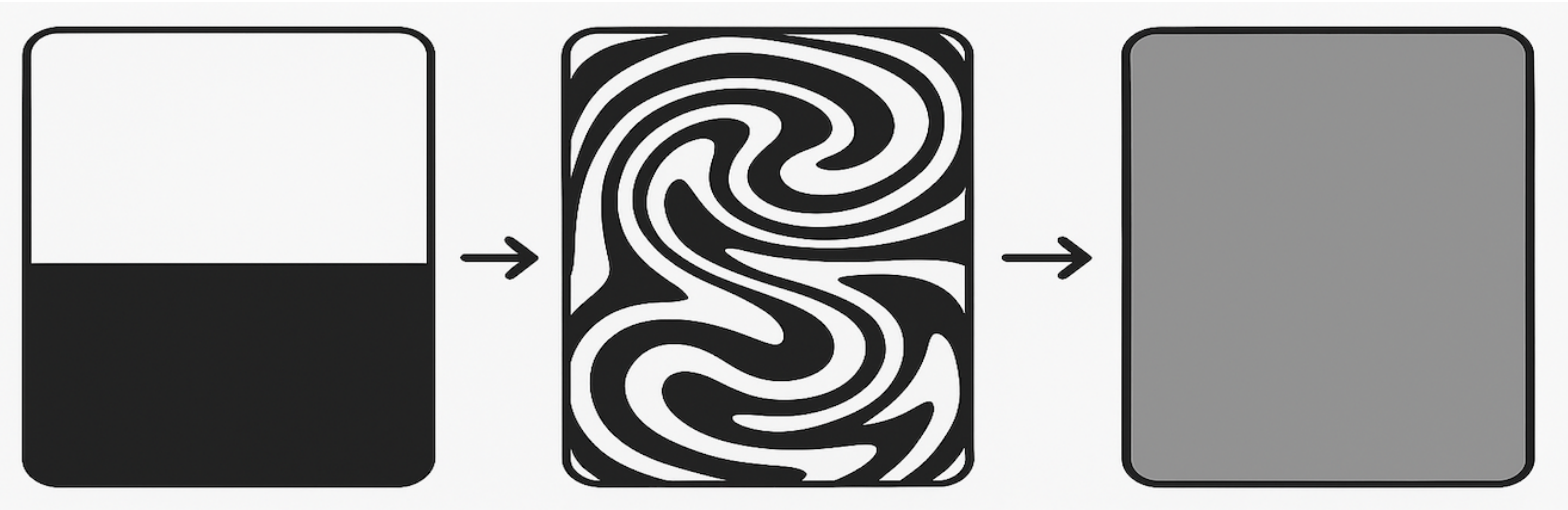

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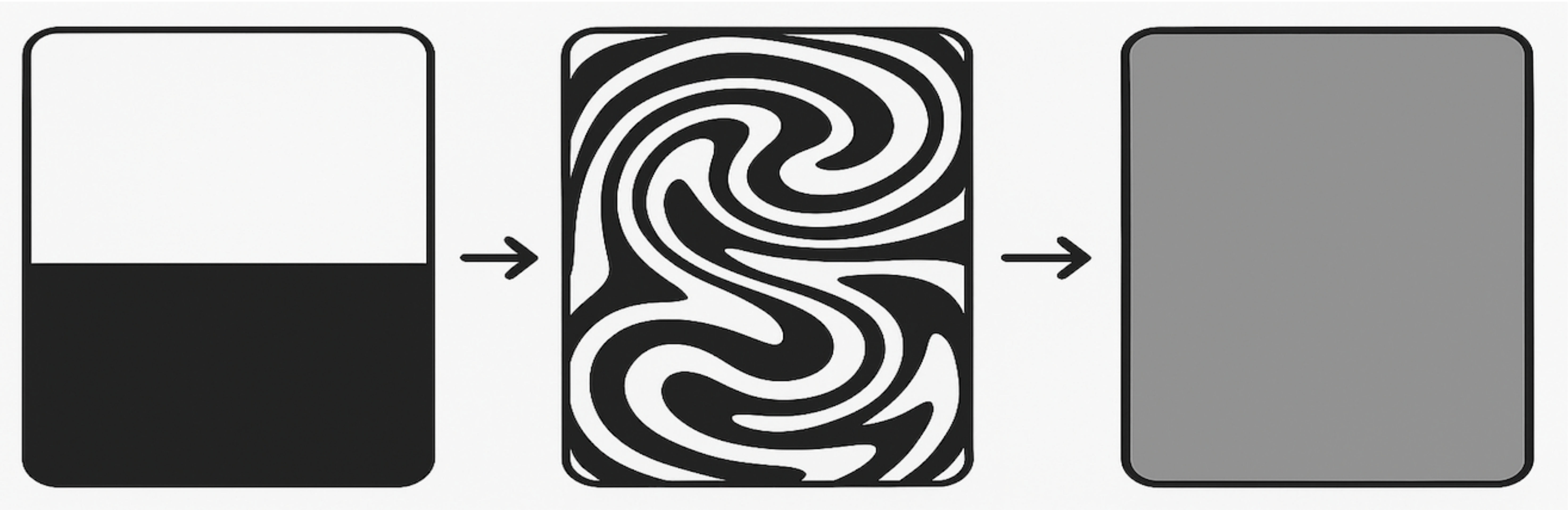

Problem Statement

$$\min_{\rho, v} \int_0^T \underbrace{\|v_t\|^2}_{\text{effort}} \, dt + \underbrace{\alpha \, d^2(\rho_T, \rho_*)}_{\text{mixedness}} \quad \text{s.t.} \quad \begin{cases} \partial_t \rho = -\nabla \cdot (\rho v) \\ \nabla \cdot v = 0 \end{cases}$$

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- ρ_* uniform $\rightarrow$ classical mixing
- Dynamic OT + incompressibility + final state penalty

Problem Decomposition (Thm 15)

$$\min_{\rho,v} \int_0^T \|v_t\|^2\,dt + \alpha\, d^2(\rho_T, \rho_*) \quad \text{s.t.} \quad \begin{cases} \partial_t \rho = -\nabla \cdot (\rho v) \\ \nabla \cdot v = 0 \\ \rho_0 \text{ fixed} \end{cases}$$

$$\Updownarrow$$

$$\min_{\rho_T} \underbrace{\left[\min_{\rho,v} \int_0^T \|v_t\|^2\,dt \quad \text{s.t.} \quad \begin{cases} \partial_t \rho = -\nabla \cdot (\rho v) \\ \nabla \cdot v = 0 \\ \rho_0, \rho_T \text{ fixed} \end{cases} \right]}_{\text{forms metric on reachable set}} + \alpha\, d^2(\rho_T, \rho_*) \quad \text{s.t.} \quad \underbrace{\rho_T \in R}_{\text{reachable set}}$$

(Geo)metric Structure

$$m^2(\rho_0, \rho_T) \quad := \quad \min_{\rho, v} \int_0^T \|v_t\|^2 \, dt \quad \text{s.t.} \quad \begin{cases} \partial_t \rho = -\nabla \cdot (\rho v) \\ \nabla \cdot v = 0 \\ \rho_0,\ \rho_T \text{ fixed} \end{cases}$$

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- Induced by Riemannian submersion on Sdiff = {volume-preserving diffeo's} (analogous result to Otto [1])

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- Optimal mixing flows = geodesics

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Geodesic Equations (Thm 20)

Supposing that $\|v\| = \|Kv\|_{L^2}$,

geodesic equations

$$\begin{cases} \partial_t \rho = -\nabla \cdot (\rho v) \\ \nabla \cdot v = 0 \\ \partial_t \lambda = -\nabla \lambda \cdot v \\ K^* K v = \rho \nabla \lambda + \nabla \gamma \end{cases} \Rightarrow \begin{cases} \partial_t \rho = -\nabla \rho \cdot v \\ \partial_t \lambda = -\nabla \lambda \cdot v \\ v = \Pi\left[(K^* K)^{-1}(\rho \nabla \lambda)\right] \end{cases}$$

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Expect Invertible

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Lagrange Multiplier

Leray Projector

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- Simultaneous transport by v
- Expect PDE for evolution of v

Geodesics in L^2

Kinetic energy penalty: $\|v\| = \|v\|_{L^2}$, $K = I$

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- Enstrophy norm $||\nabla v||_{L^2}$ likely better behaved

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- Numerical simulations

Thanks for watching! Questions?

Link to Paper