Continuum Swarm Tracking Control: A Geometric Perspective in Wasserstein Space

Max Emerick and Bassam Bamieh

University of California Santa Barbara

Conference on Decision and Control 2023

Many Applications for Autonomous Swarms

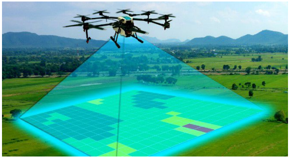
Emergency Response

Logistics

Entertainment

Transportation

Defense



Data Collection

Motivation

The Problem in Focus:

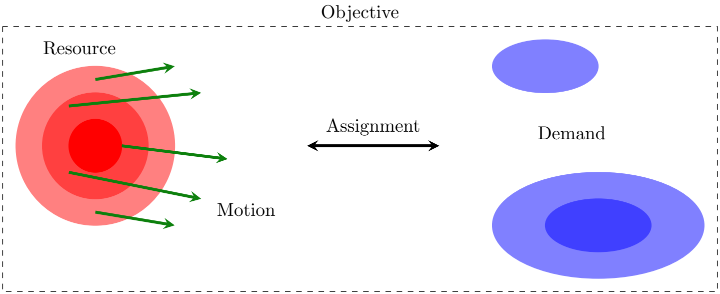
- Large swarms robust and efficient, but hard to model/control
- Want to develop theoretical foundations for design heuristics

Aim to Answer Questions:

- How should large swarms ideally move and communicate?
- Which control architectures can achieve which behavior?
- What are the attainable performance limits of these architectures?

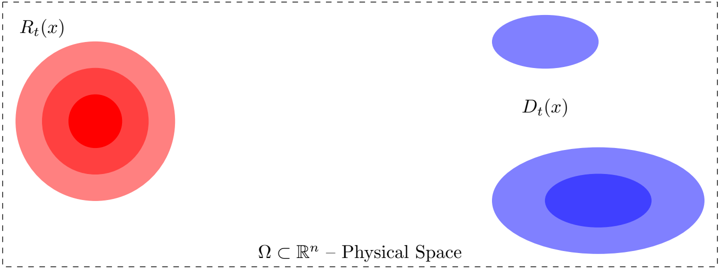
Approach

- This work: what sorts of motions patterns are optimal?
- Simplest problem first: **motion planning/control for tracking**
- Based on **continuum models**, **optimal transport**, **optimal control**



Problem Formulation: Demand/Resource Distributions

- Demand = known entity (requiring services)
- Resource = controlled mobile agents (providing services)
- In this work, we focus on continuous distributions
  ($\sim$ continuum models for large-scale swarms)

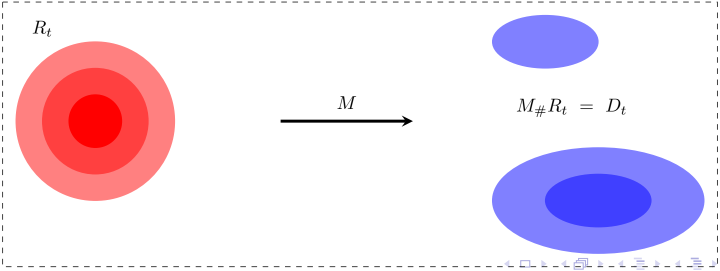


Problem Formulation: Assignment

Monge Problem (Optimal Transport)

$$\inf_{M} \int_{\Omega} |M(x) - x|^2 \, R_t(x) \, dx \qquad \text{s.t.} \qquad M_{\#} \, R_t \;=\; D_t$$

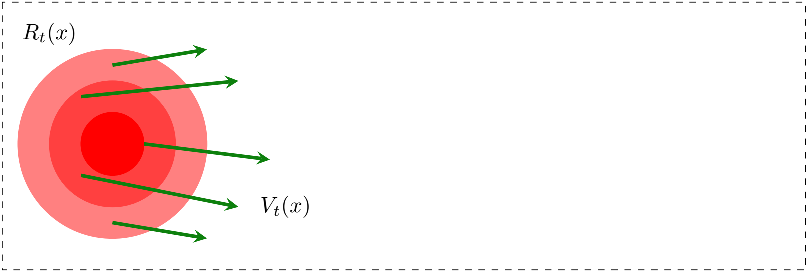
- $\#$ is the **measure pushforward**
- Minimizer $\bar{M}_{R_t \to D_t}$ is **optimal transport map**
- Minimum is **Wasserstein distance** $\mathcal{W}_2^2(R_t, D_t)$



Problem Formulation: Dynamic Model

- Tracking $\rightarrow$ want resource close to demand
- Control resource with **velocity field V**
- Dynamics given by **transport equation**:

$$\partial_t R(x,t) \;=\; -\nabla \cdot \big(V(x,t)R(x,t)\big)$$



- Motion cost : $\|V_t\|^2_{L^2(R_t)} := \int |V_t(x)|^2 R_t(x)\, dx$ $\quad$ ($\sim$ "energy cost")

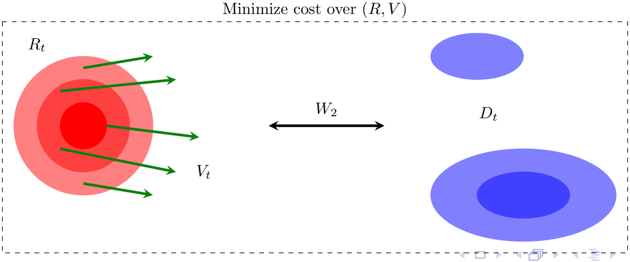
Formal Problem Statement

Proposed Problem

Given initial resource distribution R_0, demand trajectory D, solve

$$\inf_{R,V} \int_0^T \underbrace{\mathcal{W}_2^2(R,D)}_{\text{Assignment Cost}} + \alpha \underbrace{\left\|V\right\|_{L^2(R)}^2}_{\text{Motion Cost}} dt \qquad \text{s.t.} \qquad \underbrace{\partial_t R = -\nabla\cdot(VR)}_{\text{Dynamic Constraint}}.$$

- Intuitively, "R should track D as efficiently as possible"
- Trade-off parameter α controls relative importance of two costs



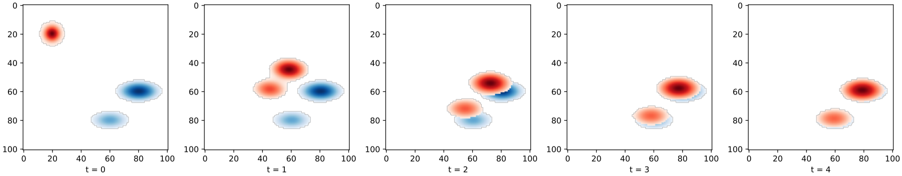
Main Result

R_0 continuous $+$ D static $\Rightarrow$

Optimal Trajectory

$$\bar{R}_t = \left[(1-\sigma(t))\,\mathcal{I} + \sigma(t)\,\bar{M}_{R_0 \to D}\right]_{\#} R_0$$

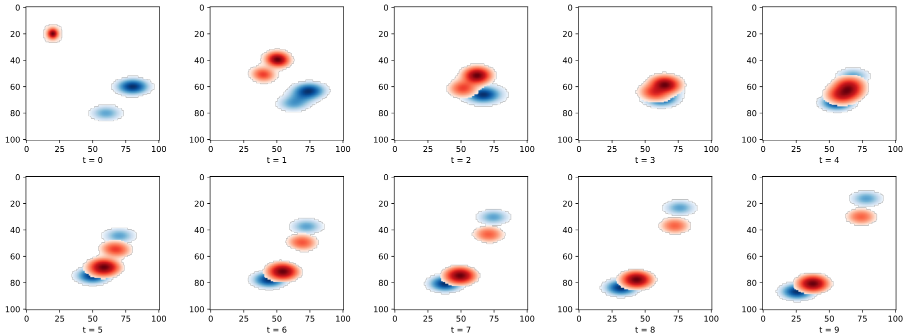
- Optimal transport tells us $\bar{R}_t$ is **geodesic** from R_0 to D
- Optimal control determines time schedule $\sigma(t)$



Practical Implications

MPC Algorithm for Time-Varying References

1. Suppose current demand distribution is static
2. Compute optimal controller, apply for τ seconds
3. Update demand and repeat



Theoretical Implications

Surprising that problem is tractable

Why does it turn out to be “nice”?

What can we learn from this?

Big Idea 1:

Problem is Simpler When Formulated Differently

Quantile Functions (1D)

Cumulative Distribution and Quantile Functions

Distribution $\rho(x)$

$$\left.\begin{array}{ll} \text{CDF} & F_\rho(x) := \int_{-\infty}^{x} \rho(\xi)\, d\xi \\ \text{Quantile} & Q_\rho(z) := \inf\{x : F_\rho(x) \geq z\} \end{array}\right\} \text{ function inverses}$$

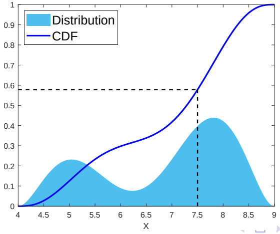
Wasserstein Distance in 1D (well-known result)

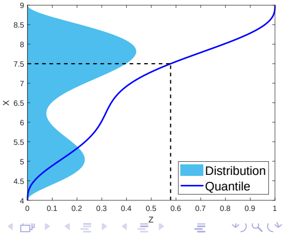
$$\mathcal{W}_2^2(\rho, \mu) \;=\; \int_0^1 \left(Q_\rho(z) - Q_\mu(z)\right)^2 dz$$

Distributions with $\mathcal{W}_2$

$\updownarrow$ isometric

Quantile functions with L^2





Equivalent Problem (1D)

Original Problem

$$\inf_{R,V} \int_0^T \mathcal{W}_2^2(R, D) \; + \; \alpha \left\| V \right\|^2_{L^2(R)} dt \qquad \text{s.t.} \qquad \partial_t R \; = \; -\nabla \cdot (VR)$$

$$\updownarrow$$

Equivalent Problem: LQ Tracking

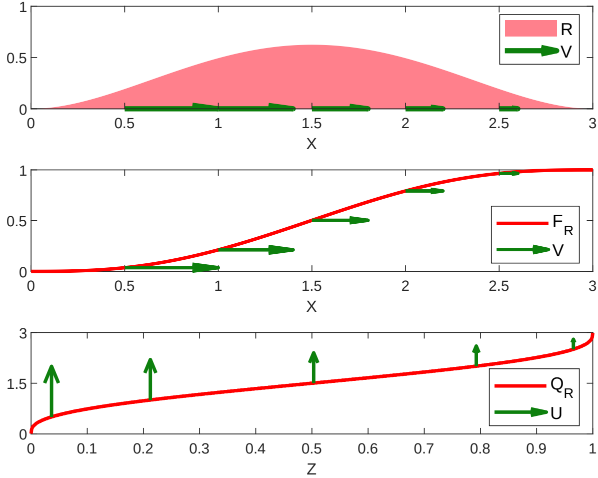
$$\inf_{Q_R, U} \int_0^T \int_0^1 \underbrace{\left(Q_R - Q_D\right)^2 \; + \; \alpha \, U^2}_{\text{quadratic cost function}} dz \, dt \qquad \text{s.t.} \qquad \underbrace{\tfrac{d}{dt} Q_R(z,t) \; = \; U(z,t)}_{\substack{\text{linear dynamics} \\ \text{(decoupled)}}}$$

- Equivalent problem has straightforward analytic solution

Equivalent Problem

Original Problem $\rightarrow$ Dynamics

Equivalent Problem $\rightarrow$ Dynamics

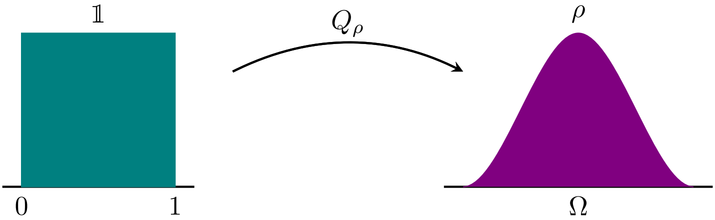


Natural Question:

Can quantile functions be extended to higher dimensions?

“Quantile Functions” in Higher Dimensions

Key Property: $\rho \;=\; [Q_\rho]_\# \mathbb{1}_{[0,1]}$



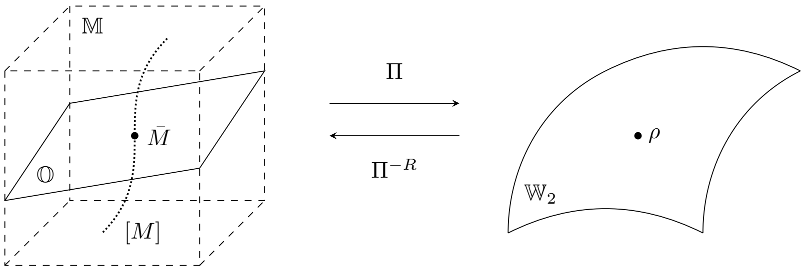
Central Idea:

- Pick reference density μ
- Represent density ρ with map M s.t. $\rho = M_\# \mu$
- Lift problem into space of maps $\mathbb{M}$

"Quantile Functions" in Higher Dimensions

- Problem: M s.t. $\rho = M_{\#}\mu$ is not unique
- Solution: use optimal transport map $\bar{M}_{\mu\to\rho}$
- OT maps have other advantages too

Densities in $\mathbb{W}_2$ 1-1 with OT maps in $\mathbb{O} \subset \mathbb{M}$:



Big Idea 2:

We can understand the geometry of $\mathbb{W}_2$ through the geometry of $\mathbb{M}$ and $\mathbb{O}$

Equivalent Geometry

- Key fact: $\Pi : \mathbb{M} \to \mathbb{W}_2$ is a **Riemannian submersion**
- ($\mathbb{M}$ and $\mathbb{W}_2$ are Riemannian manifolds, and $D\Pi$ is an orthogonal projection onto each tangent space)
- This allows us to make the following identifications:

	$\mathbb{O}$		$\mathbb{W}_2$
Points:	M		ρ
Distance:	$\lVert M - \mathcal{I} \rVert_{L^2(\mu)}$	$\longleftrightarrow$	$\mathcal{W}_2(\rho, \mu)$
Dynamics:	$\partial_t M = U$		$\partial_t \rho = -\nabla \cdot (V\rho)$
Speed:	$\lVert U \rVert_{L^2(\mu)}$		$\lVert V \rVert_{L^2(\rho)}$

$$\left(\text{Recall} \quad \lVert U \rVert_{L^2(\mu)} \;:=\; \int |U(z)|^2 \, \mu(z) \, dz \right)$$

Equivalent Problem

- What should we choose as reference density μ?
- At least when D is static, take $\mu = D$

Original Problem

$$\inf_{R,V} \int_0^T \mathcal{W}_2^2(R, D) \; + \; \alpha \left\| V \right\|_{L^2(R)}^2 dt \qquad \text{s.t.} \qquad \partial_t R \; = \; -\nabla \cdot (VR)$$

$$\updownarrow$$

Equivalent Problem: LQ Tracking

$$\inf_{M,U} \int_0^T \underbrace{\left\| M - \mathcal{I} \right\|_{L^2(D)}^2 \; + \; \alpha \left\| U \right\|_{L^2(D)}^2}_{\text{quadratic cost function}} dt \qquad \text{s.t.} \qquad \underbrace{\tfrac{d}{dt} M(z,t) \; = \; U(z,t)}_{\substack{\text{linear dynamics} \\ \text{(decoupled)}}}$$

Solution: Static Case

- Decoupling means we can solve, for each point $d \in \mathrm{supp}(D)$,

Scalar Linear-Quadratic Tracking Problem

$$\inf_{r,u} \int_0^T (r-d)^2 + \alpha u^2 \qquad \text{s.t.} \qquad \dot{r} \;=\; u,$$

and reconstruct solution to overall problem.

Solution to Scalar LQ Tracking Problem

$$\begin{aligned} &\text{Controller:} && u \;=\; -f(t)(r-d)/\alpha \\ &\text{Trajectory:} && r \;=\; \sigma(t)\, r_0 + (1-\sigma(t))\, d \\ &\text{Cost:} && \mathcal{J} \;=\; (r_0 - d)^2 \sqrt{\alpha}\, \tanh\left(T/\sqrt{\alpha}\right)/2 \end{aligned}$$

- Linear interpolation between initial state and transformed demand

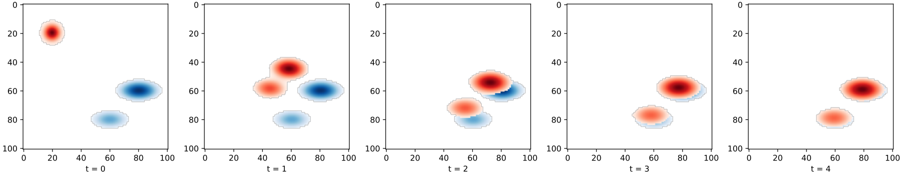
Solution: Static Case

Pulling this back to our original problem...

Solution

$$
\begin{aligned}
\text{Controller:} \quad & \bar{V}_t = -f(t)\left(\mathcal{I} - \bar{M}_{R_t \to D}\right)/\alpha \\
\text{Trajectory:} \quad & \bar{R}_t = \left[(1-\sigma(t))\,\mathcal{I} + \sigma(t)\,\bar{M}_{R_0 \to D}\right]_{\#} R_0 \\
\text{Cost:} \quad & \mathcal{J} = \mathcal{W}_2^2(R_0, D)\,\sqrt{\alpha}\,\tanh\left(T/\sqrt{\alpha}\right)/2
\end{aligned}
$$

- $\bar{R}_t$ moves along **geodesic** from R_0 to D
- Time schedule $\sigma(t)$ controlled by α, T



Takeaways

When demand is static:

- Optimal motion of the resource follows the geodesic
- Optimal motion of resource particles decouples: each resource particle only requires knowledge of its assigned demand particle

This problem is “nice” because:

- We can turn it into an equivalent problem with a lot of structure

What can we learn from this?

- Exploiting problem structure can go a long way
- Geometric structure can be very powerful

Thanks for Watching!

Questions?